

# Complex number

A **complex number** is a number that can be expressed in the form  $a + bi$ , where  $a$  and  $b$  are real numbers, and  $i$  represents the imaginary unit, satisfying the equation  $i^2 = -1$ . Because no real number satisfies this equation,  $i$  is called an imaginary number. For the complex number  $a + bi$ ,  $a$  is called the **real part**, and  $b$  is called the **imaginary part**. The set of complex numbers is denoted by either of the symbols  $\mathbb{C}$  or  $\mathbf{C}$ . Despite the historical nomenclature "imaginary", complex numbers are regarded in the mathematical sciences as just as "real" as the real numbers, and are fundamental in many aspects of the scientific description of the natural world.<sup>[note 1][1][2][3][4]</sup>

Complex numbers allow solutions to certain equations that have no solutions in real numbers. For example, the equation

$$(x + 1)^2 = -9$$

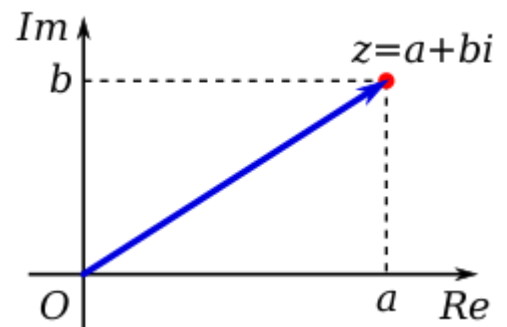
has no real solution, since the square of a real number cannot be negative. Complex numbers, however, provide a solution to this problem. The idea is to extend the real numbers with an indeterminate  $i$  (sometimes called the imaginary unit) taken to satisfy the relation  $i^2 = -1$ , so that solutions to equations like the preceding one can be found. In this case, the solutions are  $-1 + 3i$  and  $-1 - 3i$ , as can be verified using the fact that  $i^2 = -1$ :

$$\begin{aligned} ((-1 + 3i) + 1)^2 &= (3i)^2 = (3^2)(i^2) = 9(-1) = -9, \\ ((-1 - 3i) + 1)^2 &= (-3i)^2 = (-3)^2(i^2) = 9(-1) = -9. \end{aligned}$$

According to the fundamental theorem of algebra, all polynomial equations with real or complex coefficients in a single variable have a solution in complex numbers. In contrast, some polynomial equations with real coefficients have no solution in real numbers. The 16th-century Italian mathematician Gerolamo Cardano is credited with introducing complex numbers—in his attempts to find solutions to cubic equations.<sup>[5]</sup>

Formally, the complex number system can be defined as the algebraic extension of the ordinary real numbers by an imaginary number  $i$ .<sup>[6]</sup> This means that complex numbers can be added, subtracted and multiplied as polynomials in the variable  $i$ , under the rule that  $i^2 = -1$ . Furthermore, complex numbers can also be divided by nonzero complex numbers.<sup>[3]</sup> Overall, the complex number system is a field.

Geometrically, complex numbers extend the concept of the one-dimensional number line to the two-dimensional complex plane, by using the horizontal axis for the real part, and the vertical axis for the imaginary part. The complex number  $a + bi$  can be identified with the point  $(a, b)$  in the complex plane. A complex number whose real part is zero is said to be purely imaginary, and the points for these numbers lie on the vertical axis of the complex plane. Similarly, a complex number whose imaginary part is zero can be viewed as a real number, whose point lies on the horizontal axis of the complex plane. Complex numbers can also be represented in polar form, which associates each complex number with its distance from the origin (its magnitude), and a particular angle known as the argument of the complex number.



A complex number can be visually represented as a pair of numbers  $(a, b)$  forming a vector on a diagram called an Argand diagram, representing the complex plane. "Re" is the real axis, "Im" is the imaginary axis, and  $i$  satisfies  $i^2 = -1$ .

The geometric identification of the complex numbers with the complex plane, which is a Euclidean plane ( $\mathbb{R}^2$ ), makes their structure as a real 2-dimensional vector space evident. Real and imaginary parts of a complex number may be taken as components of a vector—with respect to the canonical standard basis. The addition of complex numbers is thus immediately depicted as the usual component-wise addition of vectors. However, the complex numbers allow for a richer algebraic structure, comprising additional operations, that are not necessarily available in a vector space. For example, the multiplication of two complex numbers always yields again a complex number, and should not be mistaken for the usual "products" involving vectors, like the *scalar multiplication*, the *scalar product* or other (sesqui)linear forms, available in many vector spaces; and the broadly exploited *vector product* exists only in an orientation-dependent form in three dimensions.

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## **Definition**

A complex number is a number of the form  $a + bi$ , where  $a$  and  $b$  are real numbers, and  $i$  is an indeterminate satisfying  $i^2 = -1$ . For example,  $2 + 3i$  is a complex number.<sup>[7][3]</sup>

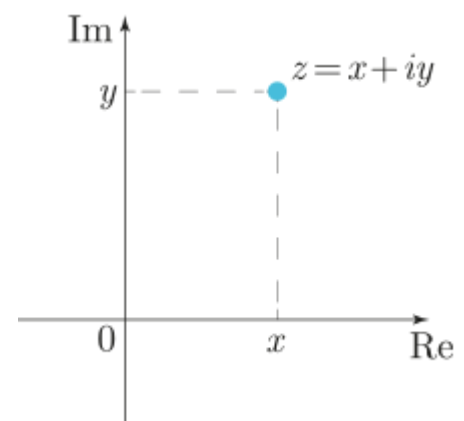
This way, a complex number is defined as a [polynomial](#) with real coefficients in the single indeterminate  $i$ , for which the relation  $i^2 + 1 = 0$  is imposed. Based on this definition, complex numbers can be added and multiplied, using the addition and multiplication for polynomials. The relation  $i^2 + 1 = 0$  induces the equalities  $i^{4k} = 1$ ,  $i^{4k+1} = i$ ,  $i^{4k+2} = -1$ , and  $i^{4k+3} = -i$ , which hold for all integers  $k$ ; these allow the reduction of any polynomial that results from the addition and multiplication of complex numbers to a linear polynomial in  $i$ , again of the form  $a + bi$  with real coefficients  $a$ ,  $b$ .

The real number  $a$  is called the *real part* of the complex number  $a + bi$ ; the real number  $b$  is called its *imaginary part*. To emphasize, the imaginary part does not include a factor  $i$ ; that is, the imaginary part is  $b$ , not  $bi$ .<sup>[8][9][3]</sup>

Formally, the complex numbers are defined as the [quotient ring](#) of the [polynomial ring](#) in the indeterminate  $i$ , by the [ideal](#) generated by the polynomial  $i^2 + 1$  (see [below](#)).<sup>[6]</sup>

## **Notation**

A real number  $a$  can be regarded as a complex number  $a + 0i$ , whose imaginary part is 0. A purely [imaginary number](#)  $bi$  is a complex number  $0 + bi$ , whose real part is zero. As with polynomials, it is common to write  $a$  for  $a + 0i$  and  $bi$  for  $0 + bi$ . Moreover, when the imaginary part is negative, that is,  $b = -|b| < 0$ , it is



An illustration of the complex number  $z = x + iy$  on the [complex plane](#). The real part is  $x$ , and its imaginary part is  $y$ .

common to write  $a - |b|i$  instead of  $a + (-|b|)i$ ; for example, for  $b = -4$ ,  $3 - 4i$  can be written instead of  $3 + (-4)i$ .

Since the multiplication of the indeterminate  $i$  and a real is commutative in polynomials with real coefficients, the polynomial  $a + bi$  may be written as  $a + ib$ . This is often expedient for imaginary parts denoted by expressions, for example, when  $b$  is a radical.<sup>[10]</sup>

The real part of a complex number  $z$  is denoted by  $\operatorname{Re}(z)$  or  $\Re(z)$ ; the imaginary part of a complex number  $z$  is denoted by  $\operatorname{Im}(z)$  or  $\Im(z)$ .<sup>[1]</sup> For example,

$$\operatorname{Re}(2 + 3i) = 2 \quad \text{and} \quad \operatorname{Im}(2 + 3i) = 3.$$

The set of all complex numbers is denoted by  $\mathbf{C}$  (upright bold) or  $\mathbb{C}$  (blackboard bold).<sup>[1]</sup>

In some disciplines, particularly in electromagnetism and electrical engineering,  $j$  is used instead of  $i$  as  $i$  is frequently used to represent electric current.<sup>[11]</sup> In these cases, complex numbers are written as  $a + bj$ , or  $a + jb$ .

## Visualization

A complex number  $z$  can thus be identified with an ordered pair  $(\operatorname{Re}(z), \operatorname{Im}(z))$  of real numbers, which in turn may be interpreted as coordinates of a point in a two-dimensional space. The most immediate space is the Euclidean plane with suitable coordinates, which is then called **complex plane** or **Argand diagram**,<sup>[12][13][14]</sup> named after Jean-Robert Argand. Another prominent space on which the coordinates may be projected is the two-dimensional surface of a sphere, which is then called Riemann sphere.

### Cartesian complex plane

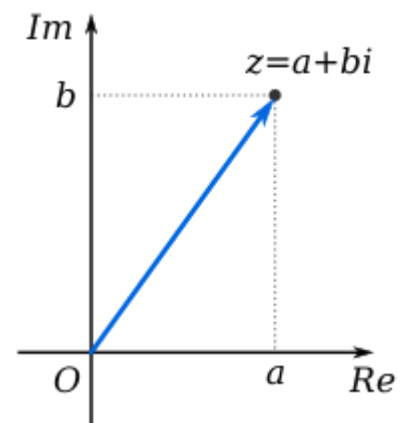
The definition of the complex numbers involving two arbitrary real values immediately suggests the use of Cartesian coordinates in the complex plane. The horizontal (*real*) axis is generally used to display the real part, with increasing values to the right, and the imaginary part marks the vertical (*imaginary*) axis, with increasing values upwards.

A charted number may be viewed either as the coordinatized point or as a position vector from the origin to this point. The coordinate values of a complex number  $z$  can hence be expressed in its *Cartesian*, *rectangular*, or *algebraic* form.

Notably, the operations of addition and multiplication take on a very natural geometric character, when complex numbers are viewed as position vectors: addition corresponds to vector addition, while multiplication (see below) corresponds to multiplying their magnitudes and adding the angles they make with the real axis. Viewed in this way, the multiplication of a complex number by  $i$  corresponds to rotating the position vector counterclockwise by a quarter turn ( $90^\circ$ ) about the origin—a fact which can be expressed algebraically as follows:

$$(a + bi) \cdot i = ai + b(i)^2 = -b + ai.$$

### Polar complex plane



A complex number  $z$ , as a point (red) and its position vector (blue)

## Modulus and argument

An alternative option for coordinates in the complex plane is the polar coordinate system that uses the distance of the point  $z$  from the origin ( $O$ ), and the angle subtended between the positive real axis and the line segment  $Oz$  in a counterclockwise sense. This leads to the polar form of complex numbers.

The absolute value (or *modulus* or *magnitude*) of a complex number  $z = x + yi$  is<sup>[15]</sup>

$$r = |z| = \sqrt{x^2 + y^2}.$$

If  $z$  is a real number (that is, if  $y = 0$ ), then  $r = |x|$ . That is, the absolute value of a real number equals its absolute value as a complex number.

By Pythagoras' theorem, the absolute value of a complex number is the distance to the origin of the point representing the complex number in the complex plane.

The argument of  $z$  (in many applications referred to as the "phase"  $\varphi$ )<sup>[14]</sup> is the angle of the radius  $Oz$  with the positive real axis, and is written as  $\arg z$ . As with the modulus, the argument can be found from the rectangular form  $x + yi$ <sup>[16]</sup>—by applying the inverse tangent to the quotient of imaginary-by-real parts. By using a half-angle identity, a single branch of the arctan suffices to cover the range of the  $\arg$ -function,  $(-\pi, \pi]$ , and avoids a more subtle case-by-case analysis

$$\varphi = \arg(x + yi) = \begin{cases} 2 \arctan\left(\frac{y}{\sqrt{x^2 + y^2} + x}\right) & \text{if } x > 0 \text{ or } y \neq 0, \\ \pi & \text{if } x < 0 \text{ and } y = 0, \\ \text{undefined} & \text{if } x = 0 \text{ and } y = 0. \end{cases}$$

Normally, as given above, the principal value in the interval  $(-\pi, \pi]$  is chosen. Values in the range  $[0, 2\pi)$  are obtained by adding  $2\pi$ —if the value is negative. The value of  $\varphi$  is expressed in radians in this article. It can increase by any integer multiple of  $2\pi$  and still give the same angle, viewed as subtended by the rays of the positive real axis and from the origin through  $z$ . Hence, the  $\arg$  function is sometimes considered as multivalued. The polar angle for the complex number 0 is indeterminate, but arbitrary choice of the polar angle 0 is common.

The value of  $\varphi$  equals the result of atan2:

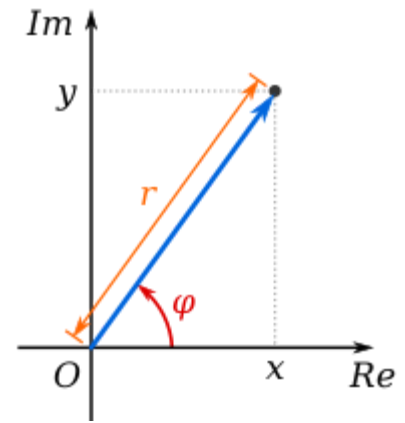
$$\varphi = \text{atan2}(\text{Im}(z), \text{Re}(z)).$$

Together,  $r$  and  $\varphi$  give another way of representing complex numbers, the *polar form*, as the combination of modulus and argument fully specify the position of a point on the plane. Recovering the original rectangular co-ordinates from the polar form is done by the formula called *trigonometric form*

$$z = r(\cos \varphi + i \sin \varphi).$$

Using Euler's formula this can be written as

$$z = re^{i\varphi}.$$



Argument  $\varphi$  and modulus  $r$  locate a point in the complex plane.

Using the cis function, this is sometimes abbreviated to

$$z = r \operatorname{cis} \varphi.$$

In angle notation, often used in electronics to represent a phasor with amplitude  $r$  and phase  $\varphi$ , it is written as<sup>[17]</sup>

$$z = r \angle \varphi.$$

## Complex graphs

When visualizing complex functions, both a complex input and output are needed. Because each complex number is represented in two dimensions, visually graphing a complex function would require the perception of a four dimensional space, which is possible only in projections. Because of this, other ways of visualizing complex functions have been designed.

In domain coloring the output dimensions are represented by color and brightness, respectively. Each point in the complex plane as domain is *ornated*, typically with *color* representing the argument of the complex number, and *brightness* representing the magnitude. Dark spots mark moduli near zero, brighter spots are farther away from the origin, the gradation may be discontinuous, but is assumed as monotonous. The colors often vary in steps of  $\frac{\pi}{3}$  for 0 to  $2\pi$  from red, yellow, green, cyan, blue, to magenta. These plots are called color wheel graphs. This provides a simple way to visualize the functions without losing information. The picture shows zeros for  $\pm 1$ ,  $(2+i)$  and poles at  $\pm\sqrt{-2-2i}$ .



A color wheel graph of the expression  $\frac{(z^2 - 1)(z - 2 - i)^2}{z^2 + 2 + 2i}$

Riemann surfaces are another way to visualize complex functions. Riemann surfaces can be thought of as deformations of the complex plane; while the horizontal axes represent the real and imaginary inputs, the single vertical axis only represents either the real or imaginary output. However, Riemann surfaces are built in such a way that rotating them 180 degrees shows the imaginary output, and vice versa. Unlike domain coloring, Riemann surfaces can represent multivalued functions like  $\sqrt{z}$ .

## History

The solution in radicals (without trigonometric functions) of a general cubic equation contains the square roots of negative numbers when all three roots are real numbers, a situation that cannot be rectified by factoring aided by the rational root test if the cubic is irreducible (the so-called *casus irreducibilis*). This conundrum led Italian mathematician Gerolamo Cardano to conceive of complex numbers in around 1545,<sup>[18]</sup> though his understanding was rudimentary.

Work on the problem of general polynomials ultimately led to the fundamental theorem of algebra, which shows that with complex numbers, a solution exists to every polynomial equation of degree one or higher. Complex numbers thus form an algebraically closed field, where any polynomial equation has a root.

Many mathematicians contributed to the development of complex numbers. The rules for addition, subtraction, multiplication, and root extraction of complex numbers were developed by the Italian mathematician Rafael Bombelli.<sup>[19]</sup> A more abstract formalism for the complex numbers was further developed by the Irish mathematician William Rowan Hamilton, who extended this abstraction to the theory of quaternions.<sup>[20]</sup>

The earliest fleeting reference to square roots of negative numbers can perhaps be said to occur in the work of the Greek mathematician Hero of Alexandria in the 1st century AD, where in his *Stereometrica* he considers, apparently in error, the volume of an impossible frustum of a pyramid to arrive at the term  $\sqrt{81 - 144} = 3i\sqrt{7}$  in his calculations, although negative quantities were not conceived of in Hellenistic mathematics and Hero merely replaced it by its positive ( $\sqrt{144 - 81} = 3\sqrt{7}$ ).<sup>[21]</sup>

The impetus to study complex numbers as a topic in itself first arose in the 16th century when algebraic solutions for the roots of cubic and quartic polynomials were discovered by Italian mathematicians (see Niccolò Fontana Tartaglia, Gerolamo Cardano). It was soon realized (but proved much later)<sup>[22]</sup> that these formulas, even if one was interested only in real solutions, sometimes required the manipulation of square roots of negative numbers. As an example, Tartaglia's formula for a cubic equation of the form  $x^3 = px + q$ <sup>[note 2]</sup> gives the solution to the equation  $x^3 = x$  as

$$\frac{1}{\sqrt{3}} \left( (\sqrt{-1})^{1/3} + (\sqrt{-1})^{-1/3} \right).$$

At first glance this looks like nonsense. However, formal calculations with complex numbers show that the equation  $z^3 = i$  has solutions  $-i$ ,  $\frac{\sqrt{3}}{2} + \frac{i}{2}$  and  $\frac{-\sqrt{3}}{2} + \frac{i}{2}$ . Substituting these in turn for  $\sqrt{-1}^{1/3}$  in Tartaglia's cubic formula and simplifying, one gets 0, 1 and  $-1$  as the solutions of  $x^3 - x = 0$ . Of course this particular equation can be solved at sight but it does illustrate that when general formulas are used to solve cubic equations with real roots then, as later mathematicians showed rigorously,<sup>[22]</sup> the use of complex numbers is unavoidable. Rafael Bombelli was the first to address explicitly these seemingly paradoxical solutions of cubic equations and developed the rules for complex arithmetic trying to resolve these issues.

The term "imaginary" for these quantities was coined by René Descartes in 1637, although he was at pains to stress their imaginary nature<sup>[23]</sup>

[...] sometimes only imaginary, that is one can imagine as many as I said in each equation, but sometimes there exists no quantity that matches that which we imagine. ([...] *quelquefois seulement imaginaires c'est-à-dire que l'on peut toujours en imaginer autant que j'ai dit en chaque équation, mais qu'il n'y a quelquefois aucune quantité qui corresponde à celle qu'on imagine.*)

A further source of confusion was that the equation  $\sqrt{-1}^2 = \sqrt{-1}\sqrt{-1} = -1$  seemed to be capriciously inconsistent with the algebraic identity  $\sqrt{a}\sqrt{b} = \sqrt{ab}$ , which is valid for non-negative real numbers  $a$  and  $b$ , and which was also used in complex number calculations with one of  $a$ ,  $b$  positive and the other negative. The incorrect use of this identity (and the related identity  $\frac{1}{\sqrt{a}} = \sqrt{\frac{1}{a}}$ ) in the case when both  $a$  and  $b$  are negative even bedeviled Euler. This difficulty eventually led to the convention of using the special symbol  $i$  in place of  $\sqrt{-1}$  to guard against this mistake. Even so, Euler considered it natural to introduce students to complex numbers much earlier than we do today. In his elementary algebra text book, *Elements of Algebra*, he introduces these numbers almost at once and then uses them in a natural way throughout.

In the 18th century complex numbers gained wider use, as it was noticed that formal manipulation of complex expressions could be used to simplify calculations involving trigonometric functions. For instance, in 1730 Abraham de Moivre noted that the complicated identities relating trigonometric functions of an integer multiple of an angle to powers of trigonometric functions of that angle could be simply re-expressed by the following well-known formula which bears his name, de Moivre's formula:

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta.$$

In 1748 Leonhard Euler went further and obtained Euler's formula of complex analysis.<sup>[24]</sup>

$$\cos \theta + i \sin \theta = e^{i\theta}$$

by formally manipulating complex power series and observed that this formula could be used to reduce any trigonometric identity to much simpler exponential identities.

The idea of a complex number as a point in the complex plane (above) was first described by Caspar Wessel in 1799,<sup>[25]</sup> although it had been anticipated as early as 1685 in Wallis's A Treatise of Algebra.<sup>[26]</sup>

Wessel's memoir appeared in the Proceedings of the Copenhagen Academy but went largely unnoticed. In 1806 Jean-Robert Argand independently issued a pamphlet on complex numbers and provided a rigorous proof of the fundamental theorem of algebra.<sup>[27]</sup> Carl Friedrich Gauss had earlier published an essentially topological proof of the theorem in 1797 but expressed his doubts at the time about "the true metaphysics of the square root of  $-1$ ".<sup>[28]</sup> It was not until 1831 that he overcame these doubts and published his treatise on complex numbers as points in the plane,<sup>[29][30]</sup> largely establishing modern notation and terminology.

If one formerly contemplated this subject from a false point of view and therefore found a mysterious darkness, this is in large part attributable to clumsy terminology. Had one not called  $+1$ ,  $-1$ ,  $\sqrt{-1}$  positive, negative, or imaginary (or even impossible) units, but instead, say, direct, inverse, or lateral units, then there could scarcely have been talk of such darkness. - Gauss<sup>[29][30]</sup>

In the beginning of the 19th century, other mathematicians discovered independently the geometrical representation of the complex numbers: Buée,<sup>[31][32]</sup> Mourey,<sup>[33]</sup> Warren,<sup>[34]</sup> Français and his brother, Bellavitis.<sup>[35][36]</sup>

The English mathematician G.H. Hardy remarked that Gauss was the first mathematician to use complex numbers in 'a really confident and scientific way' although mathematicians such as Niels Henrik Abel and Carl Gustav Jacob Jacobi were necessarily using them routinely before Gauss published his 1831 treatise.<sup>[37]</sup>

Augustin Louis Cauchy and Bernhard Riemann together brought the fundamental ideas of complex analysis to a high state of completion, commencing around 1825 in Cauchy's case.

The common terms used in the theory are chiefly due to the founders. Argand called  $\cos \varphi + i \sin \varphi$  the *direction factor*, and  $r = \sqrt{a^2 + b^2}$  the *modulus*;<sup>[38][39]</sup> Cauchy (1821) called  $\cos \varphi + i \sin \varphi$  the *reduced form* (l'expression réduite)<sup>[40]</sup> and apparently introduced the term *argument*; Gauss used  $i$  for  $\sqrt{-1}$ ,<sup>[41]</sup> introduced the term *complex number* for  $a + bi$ ,<sup>[42]</sup> and called  $a^2 + b^2$  the *norm*.<sup>[43]</sup> The expression *direction coefficient*, often used for  $\cos \varphi + i \sin \varphi$ , is due to Hankel (1867),<sup>[44]</sup> and *absolute value*, for *modulus*, is due to Weierstrass.

Later classical writers on the general theory include Richard Dedekind, Otto Hölder, Felix Klein, Henri Poincaré, Hermann Schwarz, Karl Weierstrass and many others.

## Relations and operations

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### Equality



Complex numbers have a similar definition of equality to real numbers; two complex numbers  $a_1 + b_1 i$  and  $a_2 + b_2 i$  are equal if and only if both their real and imaginary parts are equal, that is, if  $a_1 = a_2$  and  $b_1 = b_2$ . Nonzero complex numbers written in polar form are equal if and only if they have the same magnitude and their arguments differ by an integer multiple of  $2\pi$ .

## Ordering

Unlike the real numbers, there is no natural ordering of the complex numbers. In particular, there is no linear ordering on the complex numbers that is compatible with addition and multiplication – the complex numbers cannot have the structure of an ordered field. This is e.g. because every non-trivial sum of squares in an ordered field is  $\neq 0$ , and  $i^2 + 1^2 = 0$  is a non-trivial sum of squares. Thus, complex numbers are naturally thought of as existing on a two-dimensional plane.

## Conjugate

The complex conjugate of the complex number  $z = x + yi$  is given by  $x - yi$ . It is denoted by either  $\bar{z}$  or  $z^*$ .<sup>[45]</sup> This unary operation on complex numbers cannot be expressed by applying only their basic operations addition, subtraction, multiplication and division.

Geometrically,  $\bar{z}$  is the "reflection" of  $z$  about the real axis. Conjugating twice gives the original complex number

$$\overline{\bar{z}} = z,$$

which makes this operation an involution. The reflection leaves both the real part and the magnitude of  $z$  unchanged, that is

$$\operatorname{Re}(\bar{z}) = \operatorname{Re}(z) \quad \text{and} \quad |\bar{z}| = |z|.$$

The imaginary part and the argument of a complex number  $z$  change their sign under conjugation

$$\operatorname{Im}(\bar{z}) = -\operatorname{Im}(z) \quad \text{and} \quad \arg(\bar{z}) \equiv -\arg(z) \pmod{2\pi}.$$

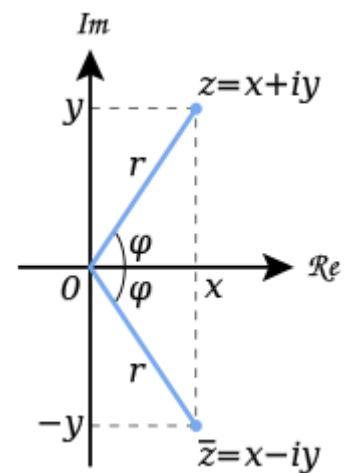
For details on argument and magnitude, see the section on Polar form.

The product of a complex number  $z = x + yi$  and its conjugate is known as the absolute square. It is always a positive real number and equals the square of the magnitude of each:

$$z \cdot \bar{z} = x^2 + y^2 = |z|^2 = |\bar{z}|^2.$$

This property can be used to convert a fraction with a complex denominator to an equivalent fraction with a real denominator by expanding both numerator and denominator of the fraction by the conjugate of the given denominator. This process is sometimes called "rationalization" of the denominator (although the denominator in the final expression might be an irrational real number), because it resembles the method to remove roots from simple expressions in a denominator.

The real and imaginary parts of a complex number  $z$  can be extracted using the conjugation:



Geometric representation of  $z$  and its conjugate  $\bar{z}$  in the complex plane

$$\operatorname{Re}(z) = \frac{z + \bar{z}}{2}, \quad \text{and} \quad \operatorname{Im}(z) = \frac{z - \bar{z}}{2i}.$$

Moreover, a complex number is real if and only if it equals its own conjugate.

Conjugation distributes over the basic complex arithmetic operations:

$$\overline{z \pm w} = \bar{z} \pm \bar{w},$$

$$\overline{z \cdot w} = \bar{z} \cdot \bar{w}, \quad \overline{z/w} = \bar{z}/\bar{w}.$$

Conjugation is also employed in inversive geometry, a branch of geometry studying reflections more general than ones about a line. In the network analysis of electrical circuits, the complex conjugate is used in finding the equivalent impedance when the maximum power transfer theorem is looked for.

## Addition and subtraction

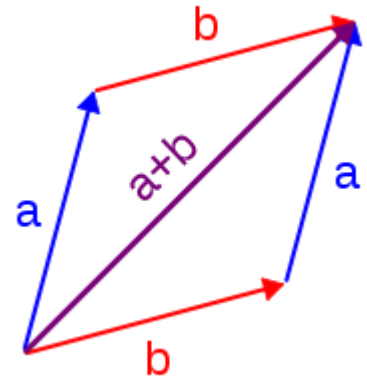
Two complex numbers  $a$  and  $b$  are most easily added by separately adding their real and imaginary parts of the summands. That is to say:

$$a + b = (x + yi) + (u + vi) = (x + u) + (y + v)i.$$

Similarly, subtraction can be performed as

$$a - b = (x + yi) - (u + vi) = (x - u) + (y - v)i.$$

Using the visualization of complex numbers in the complex plane, the addition has the following geometric interpretation: the sum of two complex numbers  $a$  and  $b$ , interpreted as points in the complex plane, is the point obtained by building a parallelogram from the three vertices  $O$ , and the points of the arrows labeled  $a$  and  $b$  (provided that they are not on a line). Equivalently, calling these points  $A$ ,  $B$ , respectively and the fourth point of the parallelogram  $X$  the triangles  $OAB$  and  $XBA$  are congruent. A visualization of the subtraction can be achieved by considering addition of the negative subtrahend.



Addition of two complex numbers can be done geometrically by constructing a parallelogram.

## Multiplication

Since the real part, the imaginary part, and the indeterminate  $i$  in a complex number are all considered as numbers in themselves, two complex numbers, given as  $z = x + yi$  and  $w = u + vi$  are multiplied under the rules of the distributive property, the commutative properties and the defining property  $i^2 = -1$  in the following way

$$\begin{aligned}
z \cdot w &= (x + yi) \cdot (u + vi) \\
&= x(u + vi) + yi(u + vi) && \text{by the (right) distributive law} \\
&= xu + xvi + yiu + yivi && \text{by the (left) distributive law} \\
&= xu + yivi + xvi + yiu && \text{by the commutativity of addition} \\
&= xu + yvi^2 + xvi + yui && \text{by the commutativity of multiplication} \\
&= (xu + yvi^2) + (xvi + yui) && \text{by the associativity of addition} \\
&= (xu - yv) + (xvi + yui) && \text{by the defining property of } i \\
&= (xu - yv) + (xv + yu)i && \text{by the distributive law.}
\end{aligned}$$

## Reciprocal and division

Using the conjugation, the reciprocal of a nonzero complex number  $z = x + yi$  can always be broken down to

$$\frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{|z|^2} = \frac{\bar{z}}{x^2 + y^2} = \frac{x}{x^2 + y^2} - \frac{y}{x^2 + y^2}i,$$

since *non-zero* implies that  $x^2 + y^2$  is greater than zero.

This can be used to express a division of an arbitrary complex number  $w = u + vi$  by a non-zero complex number  $z$  as

$$\frac{w}{z} = w \cdot \frac{1}{z} = (u + vi) \cdot \left( \frac{x}{x^2 + y^2} - \frac{y}{x^2 + y^2}i \right) = \frac{1}{x^2 + y^2} ((ux + vy) + (vx - uy)i).$$

## Multiplication and division in polar form

Formulas for multiplication, division and exponentiation are simpler in polar form than the corresponding formulas in Cartesian coordinates. Given two complex numbers  $z_1 = r_1(\cos\phi_1 + i\sin\phi_1)$  and  $z_2 = r_2(\cos\phi_2 + i\sin\phi_2)$ , because of the trigonometric identities

$$\begin{aligned}
\cos(a)\cos(b) - \sin(a)\sin(b) &= \cos(a + b) \\
\cos(a)\sin(b) + \sin(a)\cos(b) &= \sin(a + b)
\end{aligned}$$

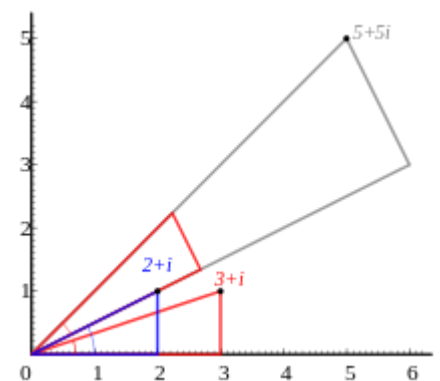
we may derive

$$z_1 z_2 = r_1 r_2 (\cos(\phi_1 + \phi_2) + i \sin(\phi_1 + \phi_2)).$$

In other words, the absolute values are multiplied and the arguments are added to yield the polar form of the product. For example, multiplying by  $i$  corresponds to a quarter-turn counter-clockwise, which gives back  $i^2 = -1$ . The picture at the right illustrates the multiplication of

$$(2 + i)(3 + i) = 5 + 5i.$$

Since the real and imaginary part of  $5 + 5i$  are equal, the argument of that number is 45 degrees, or  $\pi/4$  (in radian). On the other hand, it is also the sum of the angles at the origin of the red and blue triangles are arctan(1/3) and arctan(1/2), respectively. Thus, the formula



Multiplication of  $2 + i$  (blue triangle) and  $3 + i$  (red triangle). The red triangle is rotated to match the vertex of the blue one and stretched by  $\sqrt{5}$ , the length of the hypotenuse of the blue triangle.

$$\frac{\pi}{4} = \arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right)$$

holds. As the arctan function can be approximated highly efficiently, formulas like this – known as Machin-like formulas – are used for high-precision approximations of  $\pi$ .

Similarly, division is given by

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} (\cos(\varphi_1 - \varphi_2) + i \sin(\varphi_1 - \varphi_2)).$$

## Square root

The square roots of  $a + bi$  (with  $b \neq 0$ ) are  $\pm(\gamma + \delta i)$ , where

$$\gamma = \sqrt{\frac{a + \sqrt{a^2 + b^2}}{2}}$$

and

$$\delta = \operatorname{sgn}(b) \sqrt{\frac{-a + \sqrt{a^2 + b^2}}{2}},$$

where sgn is the signum function. This can be seen by squaring  $\pm(\gamma + \delta i)$  to obtain  $a + bi$ .<sup>[46][47]</sup> Here  $\sqrt{a^2 + b^2}$  is called the modulus of  $a + bi$ , and the square root sign indicates the square root with non-negative real part, called the **principal square root**; also  $\sqrt{a^2 + b^2} = \sqrt{z\bar{z}}$ , where  $z = a + bi$ .<sup>[48]</sup>

## Exponential function

The exponential function  $\exp: \mathbb{C} \rightarrow \mathbb{C}; z \mapsto \exp z$  can be defined for every complex number  $z$  by the power series

$$\exp z = \sum_{n=0}^{\infty} \frac{z^n}{n!},$$

which has an infinite radius of convergence.

The value at 1 of the exponential function is Euler's number

$$e = \exp 1 = \sum_{n=0}^{\infty} \frac{1}{n!} \approx 2.71828.$$

If  $z$  is real, one has  $\exp z = e^z$ . Analytic continuation allows extending this equality for every complex value of  $z$ , and thus to define the complex exponentiation with base  $e$  as

$$e^z = \exp z.$$

## Functional equation

The exponential function satisfies the functional equation  $e^{z+t} = e^z e^t$ . This can be proved either by comparing the power series expansion of both members or by applying analytic continuation from the restriction of the equation to real arguments.

## Euler's formula

Euler's formula states that, for any real number  $y$ ,

$$e^{iy} = \cos y + i \sin y.$$

The functional equation implies thus that, if  $x$  and  $y$  are real, one has

$$e^{x+iy} = e^x (\cos y + i \sin y) = e^x \cos y + i e^x \sin y,$$

which is the decomposition of the exponential function into its real and imaginary parts.

## Complex logarithm

In the real case, the natural logarithm can be defined as the inverse  $\ln: \mathbb{R}^+ \rightarrow \mathbb{R}; x \mapsto \ln x$  of the exponential function. For extending this to the complex domain, one can start from Euler's formula. It implies that, if a complex number  $z \in \mathbb{C}^\times$  is written in polar form

$$z = r(\cos \varphi + i \sin \varphi)$$

with  $r, \varphi \in \mathbb{R}$ , then with

$$\ln z = \ln r + i\varphi$$

as complex logarithm one has a proper inverse:

$$\exp \ln z = \exp(\ln r + i\varphi) = r \exp(i\varphi) = r(\cos \varphi + i \sin \varphi) = z.$$

However, because cosine and sine are periodic functions, the addition of an integer multiple of  $2\pi$  to  $\varphi$  does not change  $z$ . For example,  $e^{i\pi} = e^{3i\pi} = -1$ , so both  $i\pi$  and  $3i\pi$  are possible values for the natural logarithm of  $-1$ .

Therefore, if the complex logarithm is not to be defined as a multivalued function

$$\ln z = \{\ln r + i(\varphi + 2\pi k) \mid k \in \mathbb{Z}\},$$

one has to use a branch cut and to restrict the codomain, resulting in the bijective function

$$\ln: \mathbb{C}^\times \rightarrow \mathbb{R}^+ + i(-\pi, \pi].$$

If  $z \in \mathbb{C} \setminus (-\mathbb{R}_{\geq 0})$  is not a non-positive real number (a positive or a non-real number), the resulting principal value of the complex logarithm is obtained with  $-\pi < \varphi < \pi$ . It is an analytic function outside the negative real numbers, but it cannot be prolonged to a function that is continuous at any negative real number  $z \in -\mathbb{R}^+$ , where the principal value is  $\ln z = \ln(-z) + i\pi$ .<sup>[note 3]</sup>

# Exponentiation

If  $x > 0$  is real and  $z$  complex, the exponentiation is defined as

$$x^z = e^{z \ln x},$$

where  $\ln$  denotes the natural logarithm.

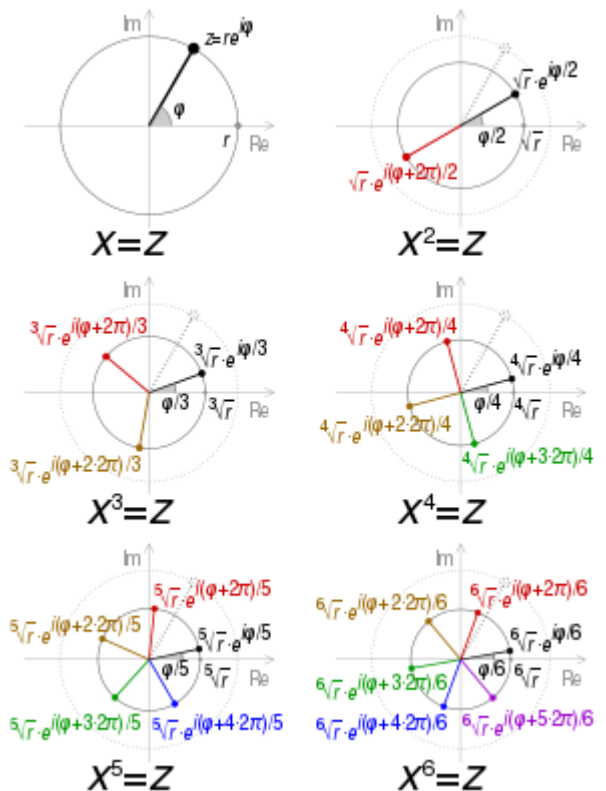
It seems natural to extend this formula to complex values of  $x$ , but there are some difficulties resulting from the fact that the complex logarithm is not really a function, but a multivalued function.

It follows that if  $z$  is as above, and if  $t$  is another complex number, then the *exponentiation* is the multivalued function

$$z^t = \{e^{t \ln r} (\cos(\varphi t + 2\pi kt) + i \sin(\varphi t + 2\pi kt))\} \mid k \in \mathbb{Z}\}$$

## Integer and fractional exponents

If, in the preceding formula,  $t$  is an integer, then the sine and the cosine are independent of  $k$ . Thus, if the exponent  $n$  is an integer, then  $z^n$  is well defined, and the exponentiation formula simplifies to de Moivre's formula:



Geometric representation of the 2nd to 6th roots of a complex number  $z$ , in polar form  $re^{i\varphi}$  where  $r = |z|$  and  $\varphi = \arg z$ . If  $z$  is real,  $\varphi = 0$  or  $\pi$ . Principal roots are shown in black.

$$z^n = (r(\cos \varphi + i \sin \varphi))^n = r^n (\cos n\varphi + i \sin n\varphi).$$

The  $n$   $n$ th roots of a complex number  $z$  are given by

$$z^{1/n} = \sqrt[n]{r} \left( \cos \left( \frac{\varphi + 2k\pi}{n} \right) + i \sin \left( \frac{\varphi + 2k\pi}{n} \right) \right)$$

for  $0 \leq k \leq n - 1$ . (Here  $\sqrt[n]{r}$  is the usual (positive)  $n$ th root of the positive real number  $r$ .) Because sine and cosine are periodic, other integer values of  $k$  do not give other values.

While the  $n$ th root of a positive real number  $r$  is chosen to be the *positive* real number  $c$  satisfying  $c^n = r$ , there is no natural way of distinguishing one particular complex  $n$ th root of a complex number. Therefore, the  $n$ th root is a  $n$ -valued function of  $z$ . This implies that, contrary to the case of positive real numbers, one has

$$(z^n)^{1/n} \neq z,$$

since the left-hand side consists of  $n$  values, and the right-hand side is a single value.

## Properties

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### Field structure

The set  $\mathbf{C}$  of complex numbers is a field.<sup>[49]</sup> Briefly, this means that the following facts hold: first, any two complex numbers can be added and multiplied to yield another complex number. Second, for any complex number  $z$ , its additive inverse  $-z$  is also a complex number; and third, every nonzero complex number has a reciprocal complex number. Moreover, these operations satisfy a number of laws, for example the law of commutativity of addition and multiplication for any two complex numbers  $z_1$  and  $z_2$ :

$$\begin{aligned} z_1 + z_2 &= z_2 + z_1, \\ z_1 z_2 &= z_2 z_1. \end{aligned}$$

These two laws and the other requirements on a field can be proven by the formulas given above, using the fact that the real numbers themselves form a field.

Unlike the reals,  $\mathbf{C}$  is not an ordered field, that is to say, it is not possible to define a relation  $z_1 < z_2$  that is compatible with the addition and multiplication. In fact, in any ordered field, the square of any element is necessarily positive, so  $i^2 = -1$  precludes the existence of an ordering on  $\mathbf{C}$ .<sup>[50]</sup>

When the underlying field for a mathematical topic or construct is the field of complex numbers, the topic's name is usually modified to reflect that fact. For example: complex analysis, complex matrix, complex polynomial, and complex Lie algebra.

### Solutions of polynomial equations

Given any complex numbers (called coefficients)  $a_0, \dots, a_n$ , the equation

$$a_n z^n + \dots + a_1 z + a_0 = 0$$

has at least one complex solution  $z$ , provided that at least one of the higher coefficients  $a_1, \dots, a_n$  is nonzero.<sup>[51]</sup> This is the statement of the *fundamental theorem of algebra*, of Carl Friedrich Gauss and Jean le Rond d'Alembert. Because of this fact,  $\mathbf{C}$  is called an algebraically closed field. This property does not hold for the field of rational numbers  $\mathbf{Q}$  (the polynomial  $x^2 - 2$  does not have a rational root, since  $\sqrt{2}$  is not a rational number) nor the real numbers  $\mathbf{R}$  (the polynomial  $x^2 + a$  does not have a real root for  $a > 0$ , since the square of  $x$  is positive for any real number  $x$ ).

There are various proofs of this theorem, by either analytic methods such as Liouville's theorem, or topological ones such as the winding number, or a proof combining Galois theory and the fact that any real polynomial of *odd* degree has at least one real root.

Because of this fact, theorems that hold *for any algebraically closed field* apply to  $\mathbf{C}$ . For example, any non-empty complex square matrix has at least one (complex) eigenvalue.

## Algebraic characterization

The field  $\mathbf{C}$  has the following three properties: first, it has characteristic 0. This means that  $1 + 1 + \cdots + 1 \neq 0$  for any number of summands (all of which equal one). Second, its transcendence degree over  $\mathbf{Q}$ , the prime field of  $\mathbf{C}$ , is the cardinality of the continuum. Third, it is algebraically closed (see above). It can be shown that any field having these properties is isomorphic (as a field) to  $\mathbf{C}$ . For example, the algebraic closure of  $\mathbf{Q}_p$  also satisfies these three properties, so these two fields are isomorphic (as fields, but not as topological fields).<sup>[52]</sup> Also,  $\mathbf{C}$  is isomorphic to the field of complex Puiseux series. However, specifying an isomorphism requires the axiom of choice. Another consequence of this algebraic characterization is that  $\mathbf{C}$  contains many proper subfields that are isomorphic to  $\mathbf{C}$ .

## Characterization as a topological field

The preceding characterization of  $\mathbf{C}$  describes only the algebraic aspects of  $\mathbf{C}$ . That is to say, the properties of nearness and continuity, which matter in areas such as analysis and topology, are not dealt with. The following description of  $\mathbf{C}$  as a topological field (that is, a field that is equipped with a topology, which allows the notion of convergence) does take into account the topological properties.  $\mathbf{C}$  contains a subset  $P$  (namely the set of positive real numbers) of nonzero elements satisfying the following three conditions:

- $P$  is closed under addition, multiplication and taking inverses.
- If  $x$  and  $y$  are distinct elements of  $P$ , then either  $x - y$  or  $y - x$  is in  $P$ .
- If  $S$  is any nonempty subset of  $P$ , then  $S + P = x + P$  for some  $x$  in  $\mathbf{C}$ .

Moreover,  $\mathbf{C}$  has a nontrivial involutive automorphism  $x \mapsto x^*$  (namely the complex conjugation), such that  $x x^*$  is in  $P$  for any nonzero  $x$  in  $\mathbf{C}$ .

Any field  $F$  with these properties can be endowed with a topology by taking the sets  $B(x, p) = \{ y \mid p - (y - x)(y - x)^* \in P \}$  as a base, where  $x$  ranges over the field and  $p$  ranges over  $P$ . With this topology  $F$  is isomorphic as a topological field to  $\mathbf{C}$ .

The only connected locally compact topological fields are  $\mathbf{R}$  and  $\mathbf{C}$ . This gives another characterization of  $\mathbf{C}$  as a topological field, since  $\mathbf{C}$  can be distinguished from  $\mathbf{R}$  because the nonzero complex numbers are connected, while the nonzero real numbers are not.<sup>[53]</sup>

## Formal construction

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### Construction as ordered pairs

William Rowan Hamilton introduced the approach to define the set  $\mathbf{C}$  of complex numbers<sup>[54]</sup> as the set  $\mathbf{R}^2$  of ordered pairs  $(a, b)$  of real numbers, in which the following rules for addition and multiplication are imposed:<sup>[49]</sup>



$$(a, b) + (c, d) = (a + c, b + d)$$

$$(a, b) \cdot (c, d) = (ac - bd, bc + ad).$$

It is then just a matter of notation to express  $(a, b)$  as  $a + bi$ .

## Construction as a quotient field

Though this low-level construction does accurately describe the structure of the complex numbers, the following equivalent definition reveals the algebraic nature of  $\mathbf{C}$  more immediately. This characterization relies on the notion of fields and polynomials. A field is a set endowed with addition, subtraction, multiplication and division operations that behave as is familiar from, say, rational numbers. For example, the distributive law

$$(x + y)z = xz + yz$$

must hold for any three elements  $x$ ,  $y$  and  $z$  of a field. The set  $\mathbf{R}$  of real numbers does form a field. A polynomial  $p(X)$  with real coefficients is an expression of the form

$$a_n X^n + \dots + a_1 X + a_0,$$

where the  $a_0, \dots, a_n$  are real numbers. The usual addition and multiplication of polynomials endows the set  $\mathbf{R}[X]$  of all such polynomials with a ring structure. This ring is called the polynomial ring over the real numbers.

The set of complex numbers is defined as the quotient ring  $\mathbf{R}[X]/(X^2 + 1)$ .<sup>[6]</sup> This extension field contains two square roots of  $-1$ , namely (the cosets of)  $X$  and  $-X$ , respectively. (The cosets of)  $1$  and  $X$  form a basis of  $\mathbf{R}[X]/(X^2 + 1)$  as a real vector space, which means that each element of the extension field can be uniquely written as a linear combination in these two elements. Equivalently, elements of the extension field can be written as ordered pairs  $(a, b)$  of real numbers. The quotient ring is a field, because  $X^2 + 1$  is irreducible over  $\mathbf{R}$ , so the ideal it generates is maximal.

The formulas for addition and multiplication in the ring  $\mathbf{R}[X]$ , modulo the relation  $X^2 = -1$ , correspond to the formulas for addition and multiplication of complex numbers defined as ordered pairs. So the two definitions of the field  $\mathbf{C}$  are isomorphic (as fields).

Accepting that  $\mathbf{C}$  is algebraically closed, since it is an algebraic extension of  $\mathbf{R}$  in this approach,  $\mathbf{C}$  is therefore the algebraic closure of  $\mathbf{R}$ .

## Matrix representation of complex numbers

Complex numbers  $a + bi$  can also be represented by  $2 \times 2$  matrices that have the following form:

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

Here the entries  $a$  and  $b$  are real numbers. The sum and product of two such matrices is again of this form, and the sum and product of complex numbers corresponds to the sum and product of such matrices, the product being:

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} c & -d \\ d & c \end{pmatrix} = \begin{pmatrix} ac - bd & -ad - bc \\ bc + ad & -bd + ac \end{pmatrix}$$

The geometric description of the multiplication of complex numbers can also be expressed in terms of rotation matrices by using this correspondence between complex numbers and such matrices. Moreover, the square of the absolute value of a complex number expressed as a matrix is equal to the determinant of that matrix:

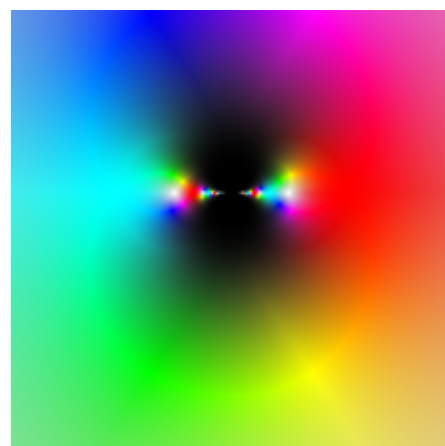
$$|z|^2 = \begin{vmatrix} a & -b \\ b & a \end{vmatrix} = a^2 + b^2.$$

The conjugate  $\bar{z}$  corresponds to the transpose of the matrix.

Though this representation of complex numbers with matrices is the most common, many other representations arise from matrices *other than*  $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  that square to the negative of the identity matrix. See the article on  $2 \times 2$  real matrices for other representations of complex numbers.

## Complex analysis

The study of functions of a complex variable is known as complex analysis and has enormous practical use in applied mathematics as well as in other branches of mathematics. Often, the most natural proofs for statements in real analysis or even number theory employ techniques from complex analysis (see prime number theorem for an example). Unlike real functions, which are commonly represented as two-dimensional graphs, complex functions have four-dimensional graphs and may usefully be illustrated by color-coding a three-dimensional graph to suggest four dimensions, or by animating the complex function's dynamic transformation of the complex plane.



Color wheel graph of  $\sin(1/z)$ . Black parts inside refer to numbers having large absolute values.

## Complex exponential and related functions

The notions of convergent series and continuous functions in (real) analysis have natural analogs in complex analysis. A sequence of complex numbers is said to converge if and only if its real and imaginary parts do. This is equivalent to the  $(\epsilon, \delta)$ -definition of limits, where the absolute value of real numbers is replaced by the one of complex numbers. From a more abstract point of view,  $\mathbb{C}$ , endowed with the metric

$$d(z_1, z_2) = |z_1 - z_2|$$

is a complete metric space, which notably includes the triangle inequality

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

for any two complex numbers  $z_1$  and  $z_2$ .

Like in real analysis, this notion of convergence is used to construct a number of elementary functions: the exponential function  $\exp z$ , also written  $e^z$ , is defined as the infinite series

$$\exp z := 1 + z + \frac{z^2}{2 \cdot 1} + \frac{z^3}{3 \cdot 2 \cdot 1} + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

The series defining the real trigonometric functions sine and cosine, as well as the hyperbolic functions  $\sinh$  and  $\cosh$ , also carry over to complex arguments without change. For the other trigonometric and hyperbolic functions, such as tangent, things are slightly more complicated, as the defining series do not converge for all

complex values. Therefore, one must define them either in terms of sine, cosine and exponential, or, equivalently, by using the method of analytic continuation.

Euler's formula states:

$$\exp(i\varphi) = \cos \varphi + i \sin \varphi$$

for any real number  $\varphi$ , in particular

$$\exp(i\pi) = -1$$

Unlike in the situation of real numbers, there is an infinitude of complex solutions  $z$  of the equation

$$\exp z = w$$

for any complex number  $w \neq 0$ . It can be shown that any such solution  $z$  – called complex logarithm of  $w$  – satisfies

$$\log w = \ln |w| + i \arg w,$$

where  $\arg$  is the argument defined above, and  $\ln$  the (real) natural logarithm. As  $\arg$  is a multivalued function, unique only up to a multiple of  $2\pi$ ,  $\log$  is also multivalued. The principal value of  $\log$  is often taken by restricting the imaginary part to the interval  $(-\pi, \pi]$ .

Complex exponentiation  $z^\omega$  is defined as

$$z^\omega = \exp(\omega \log z),$$

and is multi-valued, except when  $\omega$  is an integer. For  $\omega = 1/n$ , for some natural number  $n$ , this recovers the non-uniqueness of  $n$ th roots mentioned above.

Complex numbers, unlike real numbers, do not in general satisfy the unmodified power and logarithm identities, particularly when naïvely treated as single-valued functions; see failure of power and logarithm identities. For example, they do not satisfy

$$a^{bc} = (a^b)^c.$$

Both sides of the equation are multivalued by the definition of complex exponentiation given here, and the values on the left are a subset of those on the right.

## Holomorphic functions

A function  $f: \mathbb{C} \rightarrow \mathbb{C}$  is called holomorphic if it satisfies the Cauchy–Riemann equations. For example, any  $\mathbb{R}$ -linear map  $\mathbb{C} \rightarrow \mathbb{C}$  can be written in the form

$$f(z) = az + b\bar{z}$$

with complex coefficients  $a$  and  $b$ . This map is holomorphic if and only if  $b = 0$ . The second summand  $b\bar{z}$  is real-differentiable, but does not satisfy the Cauchy–Riemann equations.

Complex analysis shows some features not apparent in real analysis. For example, any two holomorphic functions  $f$  and  $g$  that agree on an arbitrarily small open subset of  $\mathbb{C}$  necessarily agree everywhere. Meromorphic functions, functions that can locally be written as  $f(z)/(z - z_0)^n$  with a holomorphic function  $f$ ,

still share some of the features of holomorphic functions. Other functions have essential singularities, such as  $\sin(1/z)$  at  $z = 0$ .

## Applications

Complex numbers have applications in many scientific areas, including signal processing, control theory, electromagnetism, fluid dynamics, quantum mechanics, cartography, and vibration analysis. Some of these applications are described below.

### Geometry

#### Shapes

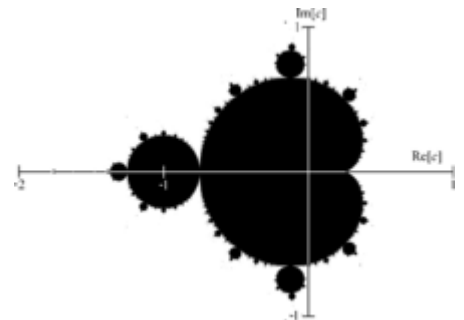
Three non-collinear points  $u, v, w$  in the plane determine the **shape** of the triangle  $\{u, v, w\}$ . Locating the points in the complex plane, this shape of a triangle may be expressed by complex arithmetic as

$$S(u, v, w) = \frac{u - w}{u - v}.$$

The shape  $S$  of a triangle will remain the same, when the complex plane is transformed by translation or dilation (by an affine transformation), corresponding to the intuitive notion of shape, and describing similarity. Thus each triangle  $\{u, v, w\}$  is in a similarity class of triangles with the same shape.<sup>[55]</sup>

#### Fractal geometry

The Mandelbrot set is a popular example of a fractal formed on the complex plane. It is defined by plotting every location  $c$  where iterating the sequence  $f_c(z) = z^2 + c$  does not diverge when iterated infinitely. Similarly, Julia sets have the same rules, except where  $c$  remains constant.



The Mandelbrot set with the real and imaginary axes labeled.

#### Triangles

Every triangle has a unique Steiner inellipse – an ellipse inside the triangle and tangent to the midpoints of the three sides of the triangle. The foci of a triangle's Steiner inellipse can be found as follows, according to Marden's theorem:<sup>[56][57]</sup> Denote the triangle's vertices in the complex plane as  $a = x_A + y_A i$ ,  $b = x_B + y_B i$ , and  $c = x_C + y_C i$ . Write the cubic equation  $(x-a)(x-b)(x-c)=0$ , take its derivative, and equate the (quadratic) derivative to zero. Marden's Theorem says that the solutions of this equation are the complex numbers denoting the locations of the two foci of the Steiner inellipse.

### Algebraic number theory

As mentioned above, any nonconstant polynomial equation (in complex coefficients) has a solution in  $\mathbb{C}$ . A fortiori, the same is true if the equation has rational coefficients. The roots of such equations are called algebraic numbers – they are a principal object of study in algebraic number theory. Compared to  $\overline{\mathbb{Q}}$ , the algebraic closure of  $\mathbb{Q}$ , which also contains all algebraic numbers,  $\mathbb{C}$  has the advantage of being easily

understandable in geometric terms. In this way, algebraic methods can be used to study geometric questions and vice versa. With algebraic methods, more specifically applying the machinery of field theory to the number field containing roots of unity, it can be shown that it is not possible to construct a regular nonagon using only compass and straightedge – a purely geometric problem.

Another example are Gaussian integers, that is, numbers of the form  $x + iy$ , where  $x$  and  $y$  are integers, which can be used to classify sums of squares.

## Analytic number theory

Analytic number theory studies numbers, often integers or rationals, by taking advantage of the fact that they can be regarded as complex numbers, in which analytic methods can be used. This is done by encoding number-theoretic information in complex-valued functions. For example, the Riemann zeta function  $\zeta(s)$  is related to the distribution of prime numbers.

Construction of a regular pentagon using straightedge and compass.

## Improper integrals

In applied fields, complex numbers are often used to compute certain real-valued improper integrals, by means of complex-valued functions. Several methods exist to do this; see methods of contour integration.

## Dynamic equations

In differential equations, it is common to first find all complex roots  $r$  of the characteristic equation of a linear differential equation or equation system and then attempt to solve the system in terms of base functions of the form  $f(t) = e^{rt}$ . Likewise, in difference equations, the complex roots  $r$  of the characteristic equation of the difference equation system are used, to attempt to solve the system in terms of base functions of the form  $f(t) = r^t$ .

## In applied mathematics

### Control theory

In control theory, systems are often transformed from the time domain to the frequency domain using the Laplace transform. The system's zeros and poles are then analyzed in the complex plane. The root locus, Nyquist plot, and Nichols plot techniques all make use of the complex plane.

In the root locus method, it is important whether zeros and poles are in the left or right half planes, that is, have real part greater than or less than zero. If a linear, time-invariant (LTI) system has poles that are

- in the right half plane, it will be unstable,
- all in the left half plane, it will be stable,
- on the imaginary axis, it will have marginal stability.

If a system has zeros in the right half plane, it is a nonminimum phase system.

## Signal analysis

Complex numbers are used in signal analysis and other fields for a convenient description for periodically varying signals. For given real functions representing actual physical quantities, often in terms of sines and cosines, corresponding complex functions are considered of which the real parts are the original quantities. For a sine wave of a given frequency, the absolute value  $|z|$  of the corresponding  $z$  is the amplitude and the argument  $\arg z$  is the phase.

If Fourier analysis is employed to write a given real-valued signal as a sum of periodic functions, these periodic functions are often written as complex valued functions of the form

$$x(t) = \operatorname{Re}\{X(t)\}$$

and

$$X(t) = Ae^{i\omega t} = ae^{i\phi}e^{i\omega t} = ae^{i(\omega t + \phi)}$$

where  $\omega$  represents the angular frequency and the complex number  $A$  encodes the phase and amplitude as explained above.

This use is also extended into digital signal processing and digital image processing, which utilize digital versions of Fourier analysis (and wavelet analysis) to transmit, compress, restore, and otherwise process digital audio signals, still images, and video signals.

Another example, relevant to the two side bands of amplitude modulation of AM radio, is:

$$\begin{aligned}\cos((\omega + \alpha)t) + \cos((\omega - \alpha)t) &= \operatorname{Re}\left(e^{i(\omega + \alpha)t} + e^{i(\omega - \alpha)t}\right) \\ &= \operatorname{Re}\left((e^{i\alpha t} + e^{-i\alpha t}) \cdot e^{i\omega t}\right) \\ &= \operatorname{Re}(2 \cos(\alpha t) \cdot e^{i\omega t}) \\ &= 2 \cos(\alpha t) \cdot \operatorname{Re}(e^{i\omega t}) \\ &= 2 \cos(\alpha t) \cdot \cos(\omega t).\end{aligned}$$

## In physics

### Electromagnetism and electrical engineering

In electrical engineering, the Fourier transform is used to analyze varying voltages and currents. The treatment of resistors, capacitors, and inductors can then be unified by introducing imaginary, frequency-dependent resistances for the latter two and combining all three in a single complex number called the impedance. This approach is called phasor calculus.

In electrical engineering, the imaginary unit is denoted by  $j$ , to avoid confusion with  $I$ , which is generally in use to denote electric current, or, more particularly,  $i$ , which is generally in use to denote instantaneous electric current.

Since the voltage in an AC circuit is oscillating, it can be represented as

$$V(t) = V_0 e^{j\omega t} = V_0 (\cos \omega t + j \sin \omega t),$$

To obtain the measurable quantity, the real part is taken:

$$v(t) = \text{Re}(V) = \text{Re} [V_0 e^{j\omega t}] = V_0 \cos \omega t.$$

The complex-valued signal  $V(t)$  is called the analytic representation of the real-valued, measurable signal  $v(t)$  .<sup>[58]</sup>

## Fluid dynamics

In fluid dynamics, complex functions are used to describe potential flow in two dimensions.

## Quantum mechanics

The complex number field is intrinsic to the mathematical formulations of quantum mechanics, where complex Hilbert spaces provide the context for one such formulation that is convenient and perhaps most standard. The original foundation formulas of quantum mechanics – the Schrödinger equation and Heisenberg's matrix mechanics – make use of complex numbers.

## Relativity

In special and general relativity, some formulas for the metric on spacetime become simpler if one takes the time component of the spacetime continuum to be imaginary. (This approach is no longer standard in classical relativity, but is used in an essential way in quantum field theory.) Complex numbers are essential to spinors, which are a generalization of the tensors used in relativity.

## Generalizations and related notions

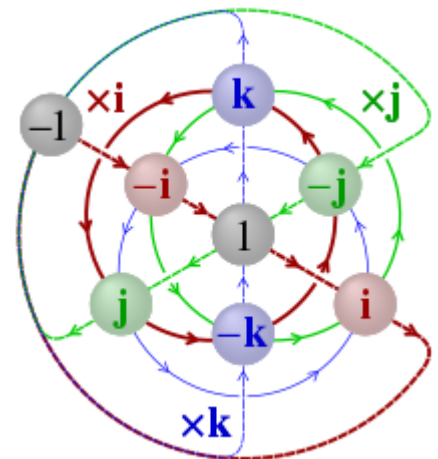
The process of extending the field  $\mathbf{R}$  of reals to  $\mathbf{C}$  is known as the Cayley–Dickson construction. It can be carried further to higher dimensions, yielding the quaternions  $\mathbf{H}$  and octonions  $\mathbf{O}$  which (as a real vector space) are of dimension 4 and 8, respectively. In this context the complex numbers have been called the **binarions**.<sup>[59]</sup>

Just as by applying the construction to reals the property of ordering is lost, properties familiar from real and complex numbers vanish with each extension. The quaternions lose commutativity, that is,  $x \cdot y \neq y \cdot x$  for some quaternions  $x, y$ , and the multiplication of octonions, additionally to not being commutative, fails to be associative:  $(x \cdot y) \cdot z \neq x \cdot (y \cdot z)$  for some octonions  $x, y, z$ .

Reals, complex numbers, quaternions and octonions are all normed division algebras over  $\mathbf{R}$ . By Hurwitz's theorem they are the only ones; the sedenions, the next step in the Cayley–Dickson construction, fail to have this structure.

The Cayley–Dickson construction is closely related to the regular representation of  $\mathbf{C}$ , thought of as an  $\mathbf{R}$ -algebra (an  $\mathbf{R}$ -vector space with a multiplication), with respect to the basis  $(1, i)$ . This means the following: the  $\mathbf{R}$ -linear map

$$\begin{aligned} \mathbb{C} &\rightarrow \mathbb{C} \\ z &\mapsto wz \end{aligned}$$



Cayley Q8 quaternion graph showing cycles of multiplication by  $i, j$  and  $k$

for some fixed complex number  $w$  can be represented by a  $2 \times 2$  matrix (once a basis has been chosen). With respect to the basis  $(1, i)$ , this matrix is

$$\begin{pmatrix} \operatorname{Re}(w) & -\operatorname{Im}(w) \\ \operatorname{Im}(w) & \operatorname{Re}(w) \end{pmatrix},$$

that is, the one mentioned in the section on matrix representation of complex numbers above. While this is a linear representation of  $\mathbf{C}$  in the  $2 \times 2$  real matrices, it is not the only one. Any matrix

$$J = \begin{pmatrix} p & q \\ r & -p \end{pmatrix}, \quad p^2 + qr + 1 = 0$$

has the property that its square is the negative of the identity matrix:  $J^2 = -I$ . Then

$$\{z = aI + bJ : a, b \in \mathbf{R}\}$$

is also isomorphic to the field  $\mathbf{C}$ , and gives an alternative complex structure on  $\mathbf{R}^2$ . This is generalized by the notion of a linear complex structure.

Hypercomplex numbers also generalize  $\mathbf{R}$ ,  $\mathbf{C}$ ,  $\mathbf{H}$ , and  $\mathbf{O}$ . For example, this notion contains the split-complex numbers, which are elements of the ring  $\mathbf{R}[x]/(x^2 - 1)$  (as opposed to  $\mathbf{R}[x]/(x^2 + 1)$ ). In this ring, the equation  $a^2 = 1$  has four solutions.

The field  $\mathbf{R}$  is the completion of  $\mathbf{Q}$ , the field of rational numbers, with respect to the usual absolute value metric. Other choices of metrics on  $\mathbf{Q}$  lead to the fields  $\mathbf{Q}_p$  of  $p$ -adic numbers (for any prime number  $p$ ), which are thereby analogous to  $\mathbf{R}$ . There are no other nontrivial ways of completing  $\mathbf{Q}$  than  $\mathbf{R}$  and  $\mathbf{Q}_p$ , by Ostrowski's theorem. The algebraic closures  $\overline{\mathbf{Q}_p}$  of  $\mathbf{Q}_p$  still carry a norm, but (unlike  $\mathbf{C}$ ) are not complete with respect to it. The completion  $\mathbf{C}_p$  of  $\overline{\mathbf{Q}_p}$  turns out to be algebraically closed. This field is called  $p$ -adic complex numbers by analogy.

The fields  $\mathbf{R}$  and  $\mathbf{Q}_p$  and their finite field extensions, including  $\mathbf{C}$ , are local fields.

## See also

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- Algebraic surface
- Circular motion using complex numbers
- Complex-base system
- Complex geometry
- Dual-complex number
- Eisenstein integer
- Euler's identity
- Geometric algebra (which includes the complex plane as the 2-dimensional spinor subspace  $\mathcal{G}_2^+$ )
- Root of unity
- Unit complex number

## Notes

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1. For an extensive account of the history, from initial skepticism to ultimate acceptance, See (Bourbaki 1998), pages 18-24.
2. In modern notation, Tartaglia's solution is based on expanding the cube of the sum of two cube roots:  $(\sqrt[3]{u} + \sqrt[3]{v})^3 = 3\sqrt[3]{uv}(\sqrt[3]{u} + \sqrt[3]{v}) + u + v$  With  $x = \sqrt[3]{u} + \sqrt[3]{v}$ ,  $p = 3\sqrt[3]{uv}$ ,  $q = u + v$ ,  $u$  and  $v$  can be expressed in terms of  $p$  and  $q$  as  $u = q/2 + \sqrt{(q/2)^2 - (p/3)^3}$  and  $v = q/2 - \sqrt{(q/2)^2 - (p/3)^3}$ , respectively. Therefore,  

$$x = \sqrt[3]{q/2 + \sqrt{(q/2)^2 - (p/3)^3}} + \sqrt[3]{q/2 - \sqrt{(q/2)^2 - (p/3)^3}}$$
When  $(q/2)^2 - (p/3)^3$  is negative (casus irreducibilis), the second cube root should be regarded as the complex conjugate of the first one.
3. However for another inverse function of the complex exponential function (and not the above defined principal value), the branch cut could be taken at any other ray thru the origin.

## References

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1. "Comprehensive List of Algebra Symbols" (<https://mathvault.ca/hub/higher-math/math-symbols/algebra-symbols/>). *Math Vault*. 25 March 2020. Retrieved 12 August 2020.
2. Penrose, Roger (2016). *The Road to Reality: A Complete Guide to the Laws of the Universe* (<https://books.google.com/books?id=VWTNCwAAQBAJ>) (reprinted ed.). Random House. pp. 72–73. ISBN 978-1-4464-1820-8. Extract of p. 73 (<https://books.google.com/books?id=VWTNCwAAQBAJ&pg=PA73>): "complex numbers, as much as reals, and perhaps even more, find a unity with nature that is truly remarkable. It is as though Nature herself is as impressed by the scope and consistency of the complex-number system as we are ourselves, and has entrusted to these numbers the precise operations of her world at its minutest scales."
3. "Complex Numbers" (<https://www.mathsisfun.com/numbers/complex-numbers.html>). *www.mathsisfun.com*. Retrieved 12 August 2020.
4. "Complex Numbers" (<https://brilliant.org/wiki/complex-numbers/>). *Brilliant Math & Science Wiki*. Retrieved 12 August 2020.
5. Burton, David M. (1995), *The History of Mathematics* (3rd ed.), New York: McGraw-Hill, p. 294, ISBN 978-0-07-009465-9
6. Bourbaki, Nicolas. "VIII.1". *General topology*. Springer-Verlag.
7. Axler, Sheldon (2010). *College algebra* (<https://archive.org/details/collegealgebrawi00axle>). Wiley. p. 262 (<https://archive.org/details/collegealgebrawi00axle/page/n285>).
8. Spiegel, M.R.; Lipschutz, S.; Schiller, J.J.; Spellman, D. (14 April 2009), *Complex Variables (2nd Edition)*, Schaum's Outline Series, McGraw Hill, ISBN 978-0-07-161569-3
9. Aufmann, Richard N.; Barker, Vernon C.; Nation, Richard D. (2007), "Chapter P" ([https://books.google.com/books?id=g5j-cT-vg\\_wC&pg=PA66](https://books.google.com/books?id=g5j-cT-vg_wC&pg=PA66)), *College Algebra and Trigonometry* (6 ed.), Cengage Learning, p. 66, ISBN 978-0-618-82515-8
10. See (Ahlfors 1979).
11. Brown, James Ward; Churchill, Ruel V. (1996), *Complex variables and applications* (6th ed.), New York: McGraw-Hill, p. 2, ISBN 978-0-07-912147-9, "In electrical engineering, the letter  $j$  is used instead of  $i$ ."
12. Pedoe, Dan (1988), *Geometry: A comprehensive course*, Dover, ISBN 978-0-486-65812-4

13. See (Solomentsev 2001): "The plane  $\mathbb{R}^2$  whose points are identified with the elements of  $\mathbb{C}$  is called the complex plane"... "The complete geometric interpretation of complex numbers and operations on them appeared first in the work of C. Wessel (1799). The geometric representation of complex numbers, sometimes called the "Argand diagram", came into use after the publication in 1806 and 1814 of papers by J.R. Argand, who rediscovered, largely independently, the findings of Wessel".
14. Weisstein, Eric W. "Complex Number" (<https://mathworld.wolfram.com/ComplexNumber.html>). *mathworld.wolfram.com*. Retrieved 12 August 2020.
15. See (Apostol 1981), page 18.
16. Kasana, H.S. (2005), "Chapter 1" (<https://books.google.com/books?id=rFhiJqkrALIC&pg=PA14>), *Complex Variables: Theory And Applications* (2nd ed.), PHI Learning Pvt. Ltd, p. 14, ISBN 978-81-203-2641-5
17. Nilsson, James William; Riedel, Susan A. (2008), "Chapter 9" (<https://books.google.com/books?id=sxmM8RFL99wC&pg=PA338>), *Electric circuits* (8th ed.), Prentice Hall, p. 338, ISBN 978-0-13-198925-2
18. Kline, Morris. *A history of mathematical thought, volume 1*. p. 253.
19. Katz, Victor J. (2004), "9.1.4", *A History of Mathematics, Brief Version*, Addison-Wesley, ISBN 978-0-321-16193-2
20. Hamilton, Wm. (1844). "On a new species of imaginary quantities connected with a theory of quaternions" (<https://babel.hathitrust.org/cgi/pt?id=njp.32101040410779&view=1up&seq=454>). *Proceedings of the Royal Irish Academy*. 2: 424–434.
21. Nahin, Paul J. (2007), *An Imaginary Tale: The Story of  $\sqrt{-1}$*  (<http://mathforum.org/kb/thread.jspa?forumID=149&threadID=383188&messageID=1181284>), Princeton University Press, ISBN 978-0-691-12798-9, archived (<https://web.archive.org/web/20121012090553/http://mathforum.org/kb/thread.jspa?forumID=149&threadID=383188&messageID=1181284>) from the original on 12 October 2012, retrieved 20 April 2011
22. Confalonieri, Sara (2015). *The Unattainable Attempt to Avoid the Casus Irreducibilis for Cubic Equations: Gerolamo Cardano's De Regula Aliza*. Springer. pp. 15–16 (note 26). ISBN 978-3658092757. "It has been proved that imaginary numbers have necessarily to appear in the cubic formula when the equation has three real, different roots by Pierre Laurent Wantzel in 1843, Vincenzo Mollame in 1890, Otto Hölder in 1891 and Adolf Kneser in 1892. Paolo Ruffini also provided an incomplete proof in 1799."
23. Descartes, René (1954) [1637], *La Géométrie | The Geometry of René Descartes with a facsimile of the first edition* (<https://archive.org/details/geometryofrenede00rend>), Dover Publications, ISBN 978-0-486-60068-0, retrieved 20 April 2011
24. Euler, Leonard (1748). *Introductio in Analysin Infinitorum* (<https://books.google.com/books?id=jQ1bAAAAQAAJ&pg=PA104>) [*Introduction to the Analysis of the Infinite*] (in Latin). vol. 1. Lucerne, Switzerland: Marc Michel Bosquet & Co. p. 104.
25. Wessel, Caspar (1799). "Om Directionens analytiske Betegning, et Forsøg, anvendt fornemmelig til plane og sphæriske Polygoners Oplosning" (<https://babel.hathitrust.org/cgi/pt?id=ien.35556000979690&view=1up&seq=527>) [On the analytic representation of direction, an effort applied in particular to the determination of plane and spherical polygons]. *Nye Samling af det Kongelige Danske Videnskabernes Selskabs Skrifter* [New Collection of the Writings of the Royal Danish Science Society] (in Danish). 5: 469–518.
26. Wallis, John (1685). *A Treatise of Algebra, Both Historical and Practical ...* (<https://echo.mpiwg-berlin.mpg.de/ECHODOCUView?url=/permanent/library/H3GRV5AU/pageimg&start=291&mode=imagepath&pn=291>). London, England: printed by John Playford, for Richard Davis. pp. 264–273.

27. Argand (1806). *Essai sur une manière de représenter les quantités imaginaires dans les constructions géométriques* (<http://www.bibnum.education.fr/mathematiques/geometrie/essai-sur-une-maniere-de-representer-des-quantites-imaginaires-dans-les-cons>) [*Essay on a way to represent complex quantities by geometric constructions*] (in French). Paris, France: Madame Veuve Blanc.
28. Gauss, Carl Friedrich (1799) "*Demonstratio nova theorematis omnem functionem algebraicam rationalem integram unius variabilis in factores reales primi vel secundi gradus resolvi posse.*" (<https://books.google.com/books?id=g3VaAAAACAAJ&pg=PP1#v=onepage&q&f=false>) [New proof of the theorem that any rational integral algebraic function of a single variable can be resolved into real factors of the first or second degree.] Ph.D. thesis, University of Helmstedt, (Germany). (in Latin)
29. Gauss, C.F. (1831). "Anzeige von Theoria residuorum biquadraticorum, commentatio secunda" (<https://babel.hathitrust.org/cgi/pt?id=uiug.30112110996987&view=1up&seq=644>) [Notice on the theory of biquadratic residues, second treatise]. *Göttingische gelehrte Anzeigen*: 638.
30. Ewald, William B. (1996). *From Kant to Hilbert: A Source Book in the Foundations of Mathematics* (<https://books.google.com/books?id=rykSDAAAQBAJ&pg=PA313>). 1. Oxford University Press. p. 313. ISBN 9780198505358. Retrieved 18 March 2020.
31. Adrien Quentin Buée (1745–1845): MacTutor (<https://mathshistory.st-andrews.ac.uk/Biographies/Buee/>)
32. Buée (1806). "Mémoire sur les quantités imaginaires" (<https://royalsocietypublishing.org/doi/pdf/10.1098/rstl.1806.0003>) [Memoir on imaginary quantities]. *Philosophical Transactions of the Royal Society of London* (in French). **96**: 23–88. doi:10.1098/rstl.1806.0003 (<https://doi.org/10.1098/rstl.1806.0003>). S2CID 110394048 (<https://api.semanticscholar.org/CorpusID:110394048>).
33. Mourey, C.V. (1861). *La vraies théore des quantités négatives et des quantités prétendues imaginaires* ([https://archive.org/details/bub\\_gb\\_8YxKAAAAYAAJ](https://archive.org/details/bub_gb_8YxKAAAAYAAJ)) [*The true theory of negative quantities and of alleged imaginary quantities*] (in French). Paris, France: Mallet-Bachelier. 1861 reprint of 1828 original.
34. See:
  - Warren, John (1828). *A Treatise on the Geometrical Representation of the Square Roots of Negative Quantities* (<https://archive.org/details/treatiseongeomet00warrich>). Cambridge, England: Cambridge University Press.
  - Warren, John (1829). "Consideration of the objections raised against the geometrical representation of the square roots of negative quantities" (<https://royalsocietypublishing.org/doi/pdf/10.1098/rstl.1829.0022>). *Philosophical Transactions of the Royal Society of London*. **119**: 241–254. doi:10.1098/rstl.1829.0022 (<https://doi.org/10.1098/rstl.1829.0022>). S2CID 186211638 (<https://api.semanticscholar.org/CorpusID:186211638>).
  - Warren, John (1829). "On the geometrical representation of the powers of quantities, whose indices involve the square roots of negative numbers" (<https://royalsocietypublishing.org/doi/pdf/10.1098/rstl.1829.0031>). *Philosophical Transactions of the Royal Society of London*. **119**: 339–359. doi:10.1098/rstl.1829.0031 (<https://doi.org/10.1098/rstl.1829.0031>). S2CID 125699726 (<https://api.semanticscholar.org/CorpusID:125699726>).
35. Français, J.F. (1813). "Nouveaux principes de géométrie de position, et interprétation géométrique des symboles imaginaires" ([https://babel.hathitrust.org/cgi/pt?id=uc1.\\$c126478&view=1up&seq=69](https://babel.hathitrust.org/cgi/pt?id=uc1.$c126478&view=1up&seq=69)) [New principles of the geometry of position, and geometric interpretation of complex [number] symbols]. *Annales des mathématiques pures et appliquées* (in French). **4**: 61–71.
36. Caparrini, Sandro (2000), "On the Common Origin of Some of the Works on the Geometrical Interpretation of Complex Numbers" (<https://books.google.com/books?id=voFsJ1EhCnYC&pg=PA139>), in Kim Williams (ed.), *Two Cultures* (<https://books.google.com/books?id=voFsJ1EhCnYC>), Birkhäuser, p. 139, ISBN 978-3-7643-7186-9
37. Hardy, G.H.; Wright, E.M. (2000) [1938], *An Introduction to the Theory of Numbers*, OUP Oxford, p. 189 (fourth edition), ISBN 978-0-19-921986-5

38. Argand (1814). "Reflexions sur la nouvelle théorie des imaginaires, suivies d'une application à la démonstration d'un théorème d'analyse" ([https://babel.hathitrust.org/cgi/pt?id=uc1.\\$c126479&view=1up&seq=209](https://babel.hathitrust.org/cgi/pt?id=uc1.$c126479&view=1up&seq=209)) [Reflections on the new theory of complex numbers, followed by an application to the proof of a theorem of analysis]. *Annales de mathématiques pures et appliquées* (in French). 5: 197–209. On p. 204, Argand defines the modulus of a complex number but he doesn't name it: "*Dans ce qui suit, les accents, indifféremment placés, seront employés pour indiquer la grandeur absolue des quantités qu'ils affectent; ainsi, si  $a = m + n\sqrt{-1}$ ,  $m$  et  $n$  étant réels, on devra entendre que  $a$  ou  $a' = \sqrt{m^2 + n^2}$ .*" (In what follows, accent marks, wherever they're placed, will be used to indicate the absolute size of the quantities to which they're assigned; thus if  $a = m + n\sqrt{-1}$ ,  $m$  and  $n$  being real, one should understand that  $a$  or  $a' = \sqrt{m^2 + n^2}$ .) On p. 208, Argand defines and names the *module* and the *direction factor* of a complex number: "*...  $a = \sqrt{m^2 + n^2}$  pourrait être appelé le module de  $a + b\sqrt{-1}$ , et représenterait la grandeur absolue de la ligne  $a + b\sqrt{-1}$ , tandis que l'autre facteur, dont le module est l'unité, en représenterait la direction.*" ( *...  $a = \sqrt{m^2 + n^2}$  could be called the module of  $a + b\sqrt{-1}$  and would represent the absolute size of the line  $a + b\sqrt{-1}$  [Note: Argand represented complex numbers as vectors.], whereas the other factor [namely,  $\frac{a}{\sqrt{a^2+b^2}} + \frac{b}{\sqrt{a^2+b^2}}\sqrt{-1}$ ], whose module is unity [1], would represent its direction.*)
39. Jeff Miller (21 September 1999). "MODULUS" (<https://web.archive.org/web/19991003034827/http://members.aol.com/jeff570/m.html>). *Earliest Known Uses of Some of the Words of Mathematics (M)*. Archived from the original on 3 October 1999.
40. Cauchy, Augustin Louis (1821). *Cours d'analyse de l'École royale polytechnique* (<https://archive.org/details/coursdanalysede00caucgoog/page/n209/mode/2up>) (in French). vol. 1. Paris, France: L'Imprimerie Royale. p. 183.
41. Gauss, C.F. (1831). "Theoria residuorum biquadraticorum. Commentatio secunda" (<https://babel.hathitrust.org/cgi/pt?id=mdp.39015073697180&view=1up&seq=292>) [Theory of biquadratic residues. Second memoir.]. *Commentationes Societatis Regiae Scientiarum Gottingensis Recentiores* (in Latin). 7: 89–148. From p. 96: "*Quemadmodum scilicet arithmetica sublimior in quaestionibus hactenus pertractatis inter solos numeros integros reales versatur, ita theoremata circa residua biquadratica tunc tantum in summa simplicitate ac genuina venustate resplendent, quando campus arithmeticae ad quantitates imaginarias extenditur, ita ut absque restrictione ipsius obiectum constituent numeri formae  $a + bi$ , denotantibus  $i$ , pro more quantitatem imaginariam  $\sqrt{-1}$ , atque  $a, b$  indefinite omnes numeros reales integros inter  $-\infty$  et  $+\infty$ .*" (Of course just as the higher arithmetic has been investigated so far in problems only among real integer numbers, so theorems regarding biquadratic residues then shine in greatest simplicity and genuine beauty, when the field of arithmetic is extended to *imaginary* quantities, so that, without restrictions on it, numbers of the form  $a + bi$  —  $i$  denoting by convention the imaginary quantity  $\sqrt{-1}$ , and the variables  $a, b$  [denoting] all real integer numbers between  $-\infty$  and  $+\infty$  — constitute an object.)
42. Gauss, C.F. (1831). "Theoria residuorum biquadraticorum. Commentatio secunda" (<https://babel.hathitrust.org/cgi/pt?id=mdp.39015073697180&view=1up&seq=292>) [Theory of biquadratic residues. Second memoir.]. *Commentationes Societatis Regiae Scientiarum Gottingensis Recentiores* (in Latin). 7: 89–148. From p. 96: "*Tales numeros vocabimus numeros integros complexos, ita quidem, ut reales complexis non opponantur, sed tamquam species sub his contineri censeantur.*" (We will call such numbers [namely, numbers of the form  $a + bi$ ] "*complex integer numbers*", so that real [numbers] are regarded not as the opposite of complex [numbers] but [as] a type [of number that] is, so to speak, contained within them.)

43. Gauss, C.F. (1831). "Theoria residuorum biquadraticorum. Commentatio secunda" (<https://babel.hathitrust.org/cgi/pt?id=mdp.39015073697180&view=1up&seq=294>) [Theory of biquadratic residues. Second memoir.]. *Commentationes Societatis Regiae Scientiarum Gottingensis Recentiores* (in Latin). 7: 89–148. From p. 98: "*Productum numeri complexi per numerum ipsi conjunctum utriusque normam vocamus. Pro norma itaque numeri realis, ipsius quadratum habendum est.*" (We call a "norm" the product of a complex number [e.g.,  $a + ib$ ] with its conjugate  $[a - ib]$ . Therefore the square of a real number should be regarded as its norm.)
44. Hankel, Hermann (1867). *Vorlesungen über die complexen Zahlen und ihre Functionen* (<https://books.google.com/books?id=754KAAAYAAJ&pg=PA71>) [*Lectures About the Complex Numbers and Their Functions*] (in German). vol. 1. Leipzig, [Germany]: Leopold Voss. p. 71. From p. 71: "*Wir werden den Factor ( $\cos \varphi + i \sin \varphi$ ) häufig den Richtungscoefficienten nennen.*" (We will often call the factor ( $\cos \varphi + i \sin \varphi$ ) the "coefficient of direction".)
45. For the former notation, See (Apostol 1981), pages 15–16.
46. Abramowitz, Milton; Stegun, Irene A. (1964), *Handbook of mathematical functions with formulas, graphs, and mathematical tables* (<https://books.google.com/books?id=MtU8uP7XMvoC>), Courier Dover Publications, p. 17, ISBN 978-0-486-61272-0, archived (<https://web.archive.org/web/20160423180235/https://books.google.com/books?id=MtU8uP7XMvoC>) from the original on 23 April 2016, retrieved 16 February 2016, Section 3.7.26, p. 17 ([http://www.math.sfu.ca/~cbm/aands/page\\_17.htm](http://www.math.sfu.ca/~cbm/aands/page_17.htm)) Archived ([https://web.archive.org/web/20090910094533/http://www.math.sfu.ca/~cbm/aands/page\\_17.htm](https://web.archive.org/web/20090910094533/http://www.math.sfu.ca/~cbm/aands/page_17.htm)) 10 September 2009 at the Wayback Machine
47. Cooke, Roger (2008), *Classical algebra: its nature, origins, and uses* (<https://books.google.com/books?id=IUcTsYopfHkC>), John Wiley and Sons, p. 59, ISBN 978-0-470-25952-8, archived (<https://web.archive.org/web/20160424023007/https://books.google.com/books?id=IUcTsYopfHkC>) from the original on 24 April 2016, retrieved 16 February 2016, Extract: page 59 (<https://books.google.com/books?id=IUcTsYopfHkC&pg=PA59>) Archived (<https://web.archive.org/web/20160423183239/https://books.google.com/books?id=IUcTsYopfHkC&pg=PA59>) 23 April 2016 at the Wayback Machine
48. See (Ahlfors 1979), page 3.
49. See (Apostol 1981), pages 15–16.
50. See (Apostol 1981), page 25.
51. Bourbaki, Nicolas. "VIII.1". *General topology*. Springer-Verlag.
52. Marker, David (1996), "Introduction to the Model Theory of Fields" (<https://projecteuclid.org/euclid.lnl/1235423155>), in Marker, D.; Messmer, M.; Pillay, A. (eds.), *Model theory of fields*, Lecture Notes in Logic, 5, Berlin: Springer-Verlag, pp. 1–37, ISBN 978-3-540-60741-0, MR 1477154 (<https://www.ams.org/mathscinet-getitem?mr=1477154>)
53. Bourbaki, Nicolas. "VIII.4". *General topology*. Springer-Verlag.
54. Corry, Leo (2015). *A Brief History of Numbers*. Oxford University Press. pp. 215–16.
55. Lester, J.A. (1994), "Triangles I: Shapes", *Aequationes Mathematicae*, 52: 30–54, doi:10.1007/BF01818325 (<https://doi.org/10.1007%2FBF01818325>), S2CID 121095307 (<https://api.semanticscholar.org/CorpusID:121095307>)
56. Kalman, Dan (2008a), "An Elementary Proof of Marden's Theorem" (<http://mathdl.maa.org/mathDL/22/?pa=content&sa=viewDocument&nodeId=3338&pf=1>), *American Mathematical Monthly*, 115 (4): 330–38, doi:10.1080/00029890.2008.11920532 (<https://doi.org/10.1080%2F00029890.2008.11920532>), ISSN 0002-9890 (<https://www.worldcat.org/issn/0002-9890>), S2CID 13222698 (<https://api.semanticscholar.org/CorpusID:13222698>), archived (<https://web.archive.org/web/20120308104622/http://mathdl.maa.org/mathDL/22/?pa=content&sa=viewDocument&nodeId=3338&pf=1>) from the original on 8 March 2012, retrieved 1 January 2012
57. Kalman, Dan (2008b), "The Most Marvelous Theorem in Mathematics" (<http://mathdl.maa.org/mathDL/4/?pa=content&sa=viewDocument&nodeId=1663>), *Journal of Online Mathematics and its Applications*, archived (<https://web.archive.org/web/20120208014954/http://mathdl.maa.org/mathDL/4/?pa=content&sa=viewDocument&nodeId=1663>) from the original on 8 February 2012, retrieved 1 January 2012

58. Grant, I.S.; Phillips, W.R. (2008), *Electromagnetism* (2 ed.), Manchester Physics Series, ISBN 978-0-471-92712-9
59. Kevin McCrimmon (2004) *A Taste of Jordan Algebras*, pp 64, Universitext, Springer ISBN 0-387-95447-3 MR2014924 (<https://mathscinet.ams.org/mathscinet-getitem?mr=2014924>)

## Works cited

- Ahlfors, Lars (1979), *Complex analysis* (3rd ed.), McGraw-Hill, ISBN 978-0-07-000657-7
- Apostol, Tom (1981), *Mathematical analysis*, Addison-Wesley
- Bourbaki, Nicolas (1998), "Foundations of mathematics § logic: set theory", *Elements of the history of mathematics*, Springer
- Solomentsev, E.D. (2001) [1994], "Complex number" ([https://www.encyclopediaofmath.org/index.php?title=Complex\\_number](https://www.encyclopediaofmath.org/index.php?title=Complex_number)), *Encyclopedia of Mathematics*, EMS Press

## Further reading

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- Penrose, Roger (2005), *The Road to Reality: A Complete Guide to the Laws of the Universe* ([https://archive.org/details/roadtorealitycom00penr\\_0](https://archive.org/details/roadtorealitycom00penr_0)), Alfred A. Knopf, ISBN 978-0-679-45443-4
- Derbyshire, John (2006), *Unknown Quantity: A Real and Imaginary History of Algebra* ([https://archive.org/details/isbn\\_9780309096577](https://archive.org/details/isbn_9780309096577)), Joseph Henry Press, ISBN 978-0-309-09657-7
- Needham, Tristan (1997), *Visual Complex Analysis*, Clarendon Press, ISBN 978-0-19-853447-1

## Mathematical

- Ahlfors, Lars (1979), *Complex analysis* (3rd ed.), McGraw-Hill, ISBN 978-0-07-000657-7
- Conway, John B. (1986), *Functions of One Complex Variable I*, Springer, ISBN 978-0-387-90328-6
- Joshi, Kapil D. (1989), *Foundations of Discrete Mathematics*, New York: John Wiley & Sons, ISBN 978-0-470-21152-6
- Pedoe, Dan (1988), *Geometry: A comprehensive course*, Dover, ISBN 978-0-486-65812-4
- Press, WH; Teukolsky, SA; Vetterling, WT; Flannery, BP (2007), "Section 5.5 Complex Arithmetic" (<http://apps.nrbook.com/empanel/index.html?pg=225>), *Numerical Recipes: The Art of Scientific Computing* (3rd ed.), New York: Cambridge University Press, ISBN 978-0-521-88068-8
- Solomentsev, E.D. (2001) [1994], "Complex number" ([https://www.encyclopediaofmath.org/index.php?title=Complex\\_number](https://www.encyclopediaofmath.org/index.php?title=Complex_number)), *Encyclopedia of Mathematics*, EMS Press

## Historical

- Bourbaki, Nicolas (1998), "Foundations of mathematics § logic: set theory", *Elements of the history of mathematics*, Springer
- Burton, David M. (1995), *The History of Mathematics* (3rd ed.), New York: McGraw-Hill, ISBN 978-0-07-009465-9
- Katz, Victor J. (2004), *A History of Mathematics, Brief Version*, Addison-Wesley, ISBN 978-0-321-16193-2
- Nahin, Paul J. (1998), *An Imaginary Tale: The Story of  $\sqrt{-1}$* , Princeton University Press, ISBN 978-0-691-02795-1

A gentle introduction to the history of complex numbers and the beginnings of complex analysis.

- Ebbinghaus, H. D.; Hermes, H.; Hirzebruch, F.; Koecher, M.; Mainzer, K.; Neukirch, J.; Prestel, A.; Remmert, R. (1991), *Numbers* (hardcover ed.), Springer, ISBN 978-0-387-97497-2

An advanced perspective on the historical development of the concept of number.

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