WIKIPEDIA

Octonion

In mathematics, the **octonions** are a normed division algebra over the real numbers, meaning it is a hypercomplex number system; Octonions are usually represented by the capital letter O, using boldface **O** or blackboard bold $\mathbb{O}$ (Unicode: 𝕆). Octonions have eight dimensions; twice the number of dimensions of the quaternions, of which they are an extension. They are noncommutative and nonassociative, but satisfy a weaker form of associativity; namely, they are alternative. They are also power associative.

Octonions are not as well known as the quaternions and complex numbers, which are much more widely studied and used. Octonions are related to exceptional structures in mathematics, among them the exceptional Lie groups. Octonions have applications in fields such as string theory, special relativity and quantum logic. Applying the Cayley–Dickson construction to the octonions produces the sedenions.

Contents

History

The octonions were discovered in 1843 by John T. Graves, inspired by his friend William Rowan Hamilton's discovery of quaternions. Graves called his discovery "octaves", and mentioned them in a letter to Hamilton dated 16 December 1843. He first published his result slightly later than Arthur Cayley's article.[1] The octonions were discovered independently by Cayley[2] and are sometimes referred to as "Cayley numbers" or the "Cayley algebra". Hamilton described the early history of Graves' discovery.[3]

Definition

The octonions can be thought of as octets (or 8-tuples) of real numbers. Every octonion is a real linear combination of the **unit octonions**:

$$\{e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7\},$$

where e_0 is the scalar or real element; it may be identified with the real number 1. That is, every octonion x can be written in the form

$$x = x_0 e_0 + x_1 e_1 + x_2 e_2 + x_3 e_3 + x_4 e_4 + x_5 e_5 + x_6 e_6 + x_7 e_7,$$

with real coefficients x_i.

Addition and subtraction of octonions is done by adding and subtracting corresponding terms and hence their coefficients, like quaternions. Multiplication is more complex. Multiplication is distributive over addition, so the product of two octonions can be calculated by summing the products of all the terms, again like quaternions. The product of each pair of terms can be given by multiplication of the coefficients and a multiplication table of the unit octonions, like this one (due to Cayley, 1845, and Graves, 1843):[4]

e_ie_j		e_j							
		e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_i	e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
	e_1	e_1	$-e_0$	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
	e_2	e_2	$-e_3$	$-e_0$	e_1	e_6	e_7	$-e_4$	$-e_5$
	e_3	e_3	e_2	$-e_1$	$-e_0$	e_7	$-e_6$	e_5	$-e_4$
	e_4	e_4	$-e_5$	$-e_6$	$-e_7$	$-e_0$	e_1	e_2	e_3
	e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	$-e_0$	$-e_3$	e_2
	e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	$-e_0$	$-e_1$
	e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	$-e_0$

Most off-diagonal elements of the table are antisymmetric, making it almost a skew-symmetric matrix except for the elements on the main diagonal, as well as the row and column for which e_0 is an operand.

The table can be summarized as follows:[5]

$$e_ie_j = \begin{cases} e_j, & \text{if } i = 0 \\ e_i, & \text{if } j = 0 \\ -\delta_{ij}e_0 + \varepsilon_{ijk}e_k, & \text{otherwise} \end{cases}$$

where δ_{ij} is the Kronecker delta (equal to 1 if and only if $i = j$), and ε_{ijk} is a completely antisymmetric tensor with value 1 when $ijk = 123, 145, 176, 246, 257, 347, 365$.

The above definition though is not unique, but is only one of 480 possible definitions for octonion multiplication with $e_0 = 1$. The others can be obtained by permuting and changing the signs of the non-scalar basis elements $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$. The 480 different algebras are isomorphic, and there is rarely a need to consider which particular multiplication rule is used.

Each of these 480 definitions is invariant up to signs under some 7-cycle of the points (1234567), and for each 7-cycle there are four definitions, differing by signs and reversal of order. A common choice is to use the definition invariant under the 7-cycle (1234567) with $e_1e_2 = e_4$ — by using the triangular multiplication diagram, or Fano plane, below (and instead of the multiplication table, above) — as it is particularly easy to remember the multiplication.

A variation of this sometimes used is to label the elements of the basis by the elements ∞, 0, 1, 2, ..., 6, of the projective line over the finite field of order 7. The multiplication is then given by $e_\infty = 1$ and $e_1e_2 = e_4$, and all expressions obtained from this by adding a constant (modulo 7) to all subscripts: in other words using the seven triples (124) (235) (346) (450) (561) (602) (013). These are the nonzero codewords of the quadratic residue code of length 7 over the Galois field of two elements, $GF(2)$. There is a symmetry of order 7 given by adding a constant mod 7 to all subscripts, and also a symmetry of order 3 given by multiplying all subscripts by one of the quadratic residues 1, 2, 4 mod 7.[6][7]

The multiplication table for a geometric algebra of signature $(----)$ can be given in terms of the following 7 quaternionic triples (omitting the identity element):

$$(I, j, k), (i, J, k), (i, j, K), (I, J, K), (\ast I, i, m), (\ast J, j, m), (\ast K, k, m)$$

in which the lowercase items are vectors and the uppercase ones are bivectors and $\ast = mijk$ (which is the Hodge star operator). If the $\ast$ is forced to be equal to the identity then the multiplication ceases to be associative, but the $\ast$ may be removed from the multiplication table resulting in an octonion multiplication table.

In keeping $\ast = mijk$ associative and thus not reducing the 4-dimensional geometric algebra to an octonion one, the whole multiplication table can be derived from the equation for $\ast$. Consider the gamma matrices. The formula defining the fifth gamma matrix shows that it is the $\ast$ of a four-dimensional geometric algebra of the gamma matrices.

Cayley–Dickson construction

A more systematic way of defining the octonions is via the Cayley–Dickson construction. Just as quaternions can be defined as pairs of complex numbers, the octonions can be defined as pairs of quaternions. Addition is defined pairwise. The product of two pairs of quaternions (a, b) and (c, d) is defined by

$$(a,b)(c,d) = (ac - d^*b, da + bc^*),$$

where z^* denotes the conjugate of the quaternion z. This definition is equivalent to the one given above when the eight unit octonions are identified with the pairs

$$(1, 0), (i, 0), (j, 0), (k, 0), (0, 1), (0, i), (0, j), (0, k)$$

Fano plane mnemonic

A convenient mnemonic for remembering the products of unit octonions is given by the diagram, which represents the multiplication table of Cayley and Graves.[4][9] This diagram with seven points and seven lines (the circle through 1, 2, and 3 is considered a line) is called the Fano plane. The lines are directional. The seven points correspond to the seven standard basis elements of $\operatorname{Im}(\mathbf{O})$ (see definition below). Each pair of distinct points lies on a unique line and each line runs through exactly three points.

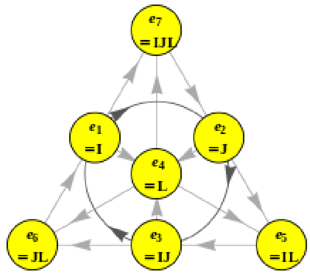
Let (a, b, c) be an ordered triple of points lying on a given line with the order specified by the direction of the arrow. Then multiplication is given by

$$ab = c \text{ and } ba = -c$$

together with cyclic permutations. These rules together with

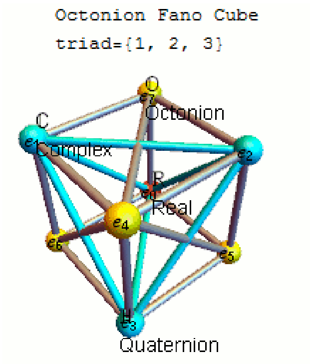
- 1 is the multiplicative identity,
- $e_i^2 = -1$ for each point in the diagram

completely defines the multiplicative structure of the octonions. Each of the seven lines generates a subalgebra of $\mathbf{O}$ isomorphic to the quaternions $\mathbf{H}$.



A mnemonic for the products of the unit octonions.[8]

Conjugate, norm, and inverse



A 3D mnemonic visualization showing the 7 triads as hyperplanes through the real (e_0) vertex of the octonion example given above.[8]

The *conjugate* of an octonion

$$x = x_0\, e_0 + x_1\, e_1 + x_2\, e_2 + x_3\, e_3 + x_4\, e_4 + x_5\, e_5 + x_6\, e_6 + x_7\, e_7$$

is given by

$$x^* = x_0\, e_0 - x_1\, e_1 - x_2\, e_2 - x_3\, e_3 - x_4\, e_4 - x_5\, e_5 - x_6\, e_6 - x_7\, e_7.$$

Conjugation is an involution of $\mathbf{O}$ and satisfies $(xy)^* = y^* x^*$ (note the change in order).

The *real part* of x is given by

$$\frac{x + x^*}{2} = x_0\, e_0$$

and the *imaginary part* by

$$\frac{x - x^*}{2} = x_1\, e_1 + x_2\, e_2 + x_3\, e_3 + x_4\, e_4 + x_5\, e_5 + x_6\, e_6 + x_7\, e_7.$$

The set of all purely imaginary octonions span a 7-dimensional subspace of $\mathbf{O}$, denoted $\mathrm{Im}(\mathbf{O})$.

Conjugation of octonions satisfies the equation

$$x^* = -\tfrac{1}{6}\big(x + (e_1 x)e_1 + (e_2 x)e_2 + (e_3 x)e_3 + (e_4 x)e_4 + (e_5 x)e_5 + (e_6 x)e_6 + (e_7 x)e_7\big).$$

The product of an octonion with its conjugate, $x^* x = x x^*$, is always a nonnegative real number:

$$x^* x = x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2.$$

Using this the norm of an octonion can be defined, as

$$\|x\| = \sqrt{x^* x}.$$

This norm agrees with the standard 8-dimensional Euclidean norm on $\mathbf{R}^8$.

The existence of a norm on $\mathbf{O}$ implies the existence of inverses for every nonzero element of $\mathbf{O}$. The inverse of $x \neq 0$ is given by

$$x^{-1} = \frac{x^*}{\|x\|^2}.$$

It satisfies $xx^{-1} = x^{-1}x = 1$.

Properties

Octonionic multiplication is neither commutative:

$$e_i e_j = -e_j e_i \neq e_j e_i \text{ if } i, j \text{ are distinct and non-zero,}$$

nor associative:

$$(e_i e_j)e_k = -e_i(e_j e_k) \neq e_i(e_j e_k) \text{ if } i, j, k \text{ are distinct, non-zero and } e_i e_j \neq \pm e_k.$$

The octonions do satisfy a weaker form of associativity: they are alternative. This means that the subalgebra generated by any two elements is associative. Actually, one can show that the subalgebra generated by any two elements of $\mathbf{O}$ is isomorphic to $\mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$, all of which are associative. Because of their non-associativity, octonions do not have matrix representations, unlike quaternions.

The octonions do retain one important property shared by $\mathbf{R}$, $\mathbf{C}$, and $\mathbf{H}$: the norm on $\mathbf{O}$ satisfies

$$\|xy\| = \|x\|\|y\|$$

This equation means that the octonions form a composition algebra. The higher-dimensional algebras defined by the Cayley–Dickson construction (starting with the sedenions) all fail to satisfy this property. They all have zero divisors.

Wider number systems exist which have a multiplicative modulus (for example, 16-dimensional conic sedenions). Their modulus is defined differently from their norm, and they also contain zero divisors.

As shown by Hurwitz, $\mathbf{R}$, $\mathbf{C}$, $\mathbf{H}$, and $\mathbf{O}$ are the only normed division algebras over the reals. These four algebras also form the only alternative, finite-dimensional division algebras over the reals (up to isomorphism).

Not being associative, the nonzero elements of $\mathbf{O}$ do not form a group. They do, however, form a loop, specifically a Moufang loop.

Commutator and cross product

The commutator of two octonions x and y is given by

$$[x, y] = xy - yx.$$

This is antisymmetric and imaginary. If it is considered only as a product on the imaginary subspace $\mathrm{Im}(\mathbf{O})$ it defines a product on that space, the seven-dimensional cross product, given by

$$x \times y = \tfrac{1}{2}(xy - yx).$$

Like the cross product in three dimensions this is a vector orthogonal to x and y with magnitude

$$\|x \times y\| = \|x\|\|y\| \sin\theta.$$

But like the octonion product it is not uniquely defined. Instead there are many different cross products, each one dependent on the choice of octonion product.[10]

Automorphisms

An automorphism, A, of the octonions is an invertible linear transformation of $\mathbf{O}$ which satisfies

$$A(xy) = A(x)A(y).$$

The set of all automorphisms of $\mathbf{O}$ forms a group called G_2.[11] The group G_2 is a simply connected, compact, real Lie group of dimension 14. This group is the smallest of the exceptional Lie groups and is isomorphic to the subgroup of $\mathrm{Spin}(7)$ that preserves any chosen particular vector in its 8-dimensional real spinor representation. The group $\mathrm{Spin}(7)$ is in turn a subgroup of the group of isotopies described below.

See also: $\mathrm{PSL}(2,7)$ – the automorphism group of the Fano plane.

Isotopies

An isotopy of an algebra is a triple of bijective linear maps a, b, c such that if $xy = z$ then $a(x)b(y) = c(z)$. For $a = b = c$ this is the same as an automorphism. The isotopy group of an algebra is the group of all isotopies, which contains the group of automorphisms as a subgroup.

The isotopy group of the octonions is the group $\mathrm{Spin}_8(\mathbf{R})$, with a, b, c acting as the three 8-dimensional representations.[12] The subgroup of elements where c fixes the identity is the subgroup $\mathrm{Spin}_7(\mathbf{R})$, and the subgroup where a, b, c all fix the identity is the automorphism group G_2.

Applications

The octonions play a significant role in the classification and construction of other mathematical entities. For example, the exceptional Lie group G_2 is the automorphism group of the octonions, and the other exceptional Lie groups F_4, E_6, E_7 and E_8 can be understood as the isometries of certain projective planes defined using the octonions.[13] The set of self-adjoint 3 × 3 octonionic matrices, equipped with a symmetrized matrix product, defines the Albert algebra. In discrete mathematics, the octonions provide an elementary derivation of the Leech lattice, and thus they are closely related to the sporadic simple groups.[14][15]

Applications of the octonions to physics have largely been conjectural. For example, in the 1970s, attempts were made to understand quarks by way of an octonionic Hilbert space.[16] It is known that the octonions, and the fact that only four normed division algebras can exist, relates to the spacetime dimensions in which supersymmetric quantum field theories can be constructed.[17][18] Also, attempts have been made to obtain the Standard Model of elementary particle physics from octonionic constructions, for example using the "Dixon algebra" $\mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$.[19][20]

Octonions have also arisen in the study of black hole entropy and quantum information science.[21][22]

Octonions have been used in solutions to the hand eye calibration problem in robotics.[23]

Deep octonion networks provide a means of efficient and compact expression in machine learning applications.[24]

Integral octonions

There are several natural ways to choose an integral form of the octonions. The simplest is just to take the octonions whose coordinates are integers. This gives a nonassociative algebra over the integers called the Gravesian octonions. However it is not a maximal order (in the sense of ring theory); there are exactly seven maximal orders containing it. These 7=seven maximal orders are all equivalent under automorphisms. The phrase "integral octonions" usually refers to a fixed choice of one of these seven orders.

These maximal orders were constructed by Kirmse (1925), Dickson and Bruck as follows. Label the eight basis vectors by the points of the projective line over the field with seven elements. First form the "Kirmse integers" : these consist of octonions whose coordinates are integers or half integers, and that are half integers (that is, halves of odd integers) on one of the 16 sets

$$\varnothing\ (\infty 124)\ (\infty 235)\ (\infty 346)\ (\infty 450)\ (\infty 561)\ (\infty 602)\ (\infty 013)\ (\infty 0123456)\ (0356)\ (1460)\ (2501)\ (3612)\ (4023)\ (5134)\ (6245)$$

of the extended quadratic residue code of length 8 over the field of two elements, given by $\varnothing$, $(\infty 124)$ and its images under adding a constant modulo 7, and the complements of these eight sets. Then switch infinity and any one other coordinate; this operation creates a bijection of the Kirmse integers onto a different set, which is a maximal order. There are seven ways to do this, giving seven maximal orders, which are all equivalent under cyclic permutations of the seven coordinates 0123456. (Kirmse incorrectly claimed that the Kirmse integers also form a maximal order, so he thought there were eight maximal orders rather than seven, but as Coxeter (1946) pointed out they are not closed under multiplication; this mistake occurs in several published papers.)

The Kirmse integers and the seven maximal orders are all isometric to the E_8 lattice rescaled by a factor of $1/\sqrt{2}$. In particular there are 240 elements of minimum nonzero norm 1 in each of these orders, forming a Moufang loop of order 240.

The integral octonions have a "division with remainder" property: given integral octonions a and $b \neq 0$, we can find q and r with $a = qb + r$, where the remainder r has norm less than that of b.

In the integral octonions, all left ideals and right ideals are 2-sided ideals, and the only 2-sided ideals are the principal ideals nO where n is a non-negative integer.

The integral octonions have a version of factorization into primes, though it is not straightforward to state because the octonions are not associative so the product of octonions depends on the order in which one does the products. The irreducible integral octonions are exactly those of prime norm, and every integral octonion can be written as a product of irreducible octonions. More precisely an integral octonion of norm mn can be written as a product of integral octonions of norms m and n.

The automorphism group of the integral octonions is the group $G_2(\mathbf{F}_2)$ of order 12 096, which has a simple subgroup of index 2 isomorphic to the unitary group ${}^2A_2(3^2)$. The isotopy group of the integral octonions is the perfect double cover of the group of rotations of the E_8 lattice.

See also

- Octonion algebra
- Okubo algebra
- $\mathrm{Spin}(8)$
- Split-octonions
- Triality

Notes

1. (Graves 1845)
2. Cayley, Arthur (1845), "On Jacobi's elliptic functions, in reply to the Rev.; and on quaternions" (https://zenodo.org/record/1431049), *Philosophical Magazine*, **26**: 208–211, doi:10.1080/14786444508645107 (https://doi.org/10.1080%2F14786444508645107). Appendix reprinted in *The Collected Mathematical Papers*, Johnson Reprint Co., New York, 1963, p. 127
3. Hamilton (1848), "Note, by Sir W. R. Hamilton, respecting the researches of John T. Graves, Esq." (https://archive.org/details/transactionsofro21iris), *Transactions of the Royal Irish Academy*, **21**: 338–341
4. G Gentili, C Stoppato, DC Struppa and F Vlacci (2009), "Recent developments for regular functions of a hypercomplex variable" (https://books.google.com/books?id=H-5v6pPpyb4C&pg=PA168), in Irene Sabadini; M Shapiro; F Sommen (eds.), *Hypercomplex analysis* (Conference on quaternionic and Clifford analysis; proceedings ed.), Birkhäuser, p. 168, ISBN 978-3-7643-9892-7
5. L. V. Sabinin; L. Sbitneva; I. P. Shestakov (2006), "§17.2 Octonion algebra and its regular bimodule representation" (https://books.google.com/books?id=_PEWt18egGgC&pg=PA235), *Non-associative algebra and its applications*, CRC Press, p. 235, ISBN 0-8247-2669-3
6. Rafał Abłamowicz; Pertti Lounesto; Josep M. Parra (1996), "§ Four ocotonionic basis numberings" (https://books.google.com/books?id=OpbY_abijtwC&pg=PA202), *Clifford algebras with numeric and symbolic computations*, Birkhäuser, p. 202, ISBN 0-8176-3907-1
7. Jörg Schray; Corinne A. Manogue (1996), "Octonionic representations of Clifford algebras and triality", *Foundations of Physics*, **26** (Number 1/January): 17–70, arXiv:hep-th/9407179 (https://arxiv.org/abs/hep-th/9407179), Bibcode:1996FoPh...26...17S (https://ui.adsabs.harvard.edu/abs/1996FoPh...26...17S), doi:10.1007/BF02058887 (https://doi.org/10.1007%2FBF02058887), S2CID 119604596 (https://api.semanticscholar.org/CorpusID:119604596). Available as ArXive preprint (https://arxiv.org/abs/hep-th/9407179v1) Figure 1 is located here (https://arxiv.org/PS_cache/hep-th/ps/9407/9407179v1.fig1-1.png).
8. (Baez 2002, p. 6)
9. Tevian Dray & Corinne A Manogue (2004), "Chapter 29: Using octonions to describe fundamental particles" (https://books.google.com/books?id=b6mbSCv_MHMC&pg=PA452), in Pertti Lounesto & Rafał Abłamowicz (eds.), *Clifford algebras: applications to mathematics, physics, and engineering*, Birkhäuser, p. 452, ISBN 0-8176-3525-4 Figure 29.1: Representation of multiplication table on projective plane.
10. (Baez 2002, pp. 37–38)
11. (Conway & Smith 2003, Chapter 8.6)
12. (Conway & Smith 2003, Chapter 8)
13. Baez (2002), section 4.
14. Wilson, Robert A. (2009-09-15). "Octonions and the Leech lattice" (http://www.maths.qmul.ac.uk/%7Eraw/pubs_files/octoLeech1rev.pdf) (PDF). *Journal of Algebra*. **322** (6): 2186–2190. doi:10.1016/j.jalgebra.2009.03.021 (https://doi.org/10.1016%2Fj.jalgebra.2009.03.021).

15. Wilson, Robert A. (2010-08-13). "Conway's group and octonions" (http://www.maths.qmul.ac.uk/~raw/pubs_files/octoConway.pdf) (PDF). *Journal of Group Theory*. **14**: 1–8. doi:10.1515/jgt.2010.038 (https://doi.org/10.1515%2Fjgt.2010.038). S2CID 16590883 (https://api.semanticscholar.org/CorpusID:16590883).
16. Günaydin, M.; Gürsey, F. (1973). "Quark structure and octonions". *Journal of Mathematical Physics*. **14** (11): 1651–1667. doi:10.1063/1.1666240 (https://doi.org/10.1063%2F1.1666240).
Günaydin, M.; Gürsey, F. (1974). "Quark statistics and octonions". *Physical Review D*. **9** (12): 3387–3391. doi:10.1103/PhysRevD.9.3387 (https://doi.org/10.1103%2FPhysRevD.9.3387).
17. Kugo, Taichiro; Townsend, Paul (1983-07-11). "Supersymmetry and the division algebras" (http://cds.cern.ch/record/140183). *Nuclear Physics B*. **221** (2): 357–380. doi:10.1016/0550-3213(83)90584-9 (https://doi.org/10.1016%2F0550-3213%2883%2990584-9).
18. Baez, John C.; Huerta, John (2010). "Division Algebras and Supersymmetry I". In Doran, R.; Friedman, G.; Rosenberg, J. (eds.). *Superstrings, Geometry, Topology, and C*-algebras*. American Mathematical Society. arXiv:0909.0551 (https://arxiv.org/abs/0909.0551).
19. Wolchover, Natalie (2018-07-20). "The Peculiar Math That Could Underlie the Laws of Nature" (https://www.quantamagazine.org/the-octonion-math-that-could-underpin-physics-20180720/). *Quanta Magazine*. Retrieved 2018-10-30.
20. Furey, Cohl (2012-07-20). "Unified theory of ideals". *Physical Review D*. **86** (2): 025024. arXiv:1002.1497 (https://arxiv.org/abs/1002.1497). Bibcode:2012PhRvD..86b5024F (https://ui.adsabs.harvard.edu/abs/2012PhRvD..86b5024F). doi:10.1103/PhysRevD.86.025024 (https://doi.org/10.1103%2FPhysRevD.86.025024). S2CID 118458623 (https://api.semanticscholar.org/CorpusID:118458623).
Furey, Cohl (2018-10-10). "Three generations, two unbroken gauge symmetries, and one eight-dimensional algebra". *Physics Letters B*. **785**: 84–89. arXiv:1910.08395 (https://arxiv.org/abs/1910.08395). Bibcode:2018PhLB..785...84F (https://ui.adsabs.harvard.edu/abs/2018PhLB..785...84F). doi:10.1016/j.physletb.2018.08.032 (https://doi.org/10.1016%2Fj.physletb.2018.08.032). S2CID 126205768 (https://api.semanticscholar.org/CorpusID:126205768).
Stoica, O. C. (2018). "Leptons, quarks, and gauge from the complex Clifford algebra $\mathbb{C}\ell_6$". *Advances in Applied Clifford Algebras*. **28**: 52. arXiv:1702.04336 (https://arxiv.org/abs/1702.04336). doi:10.1007/s00006-018-0869-4 (https://doi.org/10.1007%2Fs00006-018-0869-4). S2CID 125913482 (https://api.semanticscholar.org/CorpusID:125913482).
Gresnigt, Niels G. (2017-11-21). *Quantum groups and braid groups as fundamental symmetries*. European Physical Society conference on High Energy Physics, 5–12 July 2017, Venice, Italy. arXiv:1711.09011 (https://arxiv.org/abs/1711.09011).
Dixon, Geoffrey M. (1994). *Division Algebras: Octonions, Quaternions, Complex Numbers and the Algebraic Design of Physics*. Springer-Verlag. doi:10.1007/978-1-4757-2315-1 (https://doi.org/10.1007%2F978-1-4757-2315-1). ISBN 978-0-7923-2890-2. OCLC 30399883 (https://www.worldcat.org/oclc/30399883).
Baez, John C. (2011-01-29). "The Three-Fold Way (Part 4)" (https://golem.ph.utexas.edu/category/2011/01/the_threefold_way_part_4_1.html). *The n-Category Café*. Retrieved 2018-11-02.
21. Borsten, Leron; Dahanayake, Duminda; Duff, Michael J.; Ebrahim, Hajar; Rubens, Williams (2009). "Black holes, qubits and octonions". *Physics Reports*. **471** (3–4): 113–219. arXiv:0809.4685 (https://arxiv.org/abs/0809.4685). doi:10.1016/j.physrep.2008.11.002 (https://doi.org/10.1016%2Fj.physrep.2008.11.002). S2CID 118488578 (https://api.semanticscholar.org/CorpusID:118488578).
22. Stacey, Blake C. (2017). "Sporadic SICs and the Normed Division Algebras". *Foundations of Physics*. **47** (8): 1060–1064. arXiv:1605.01426 (https://arxiv.org/abs/1605.01426). Bibcode:2017FoPh...47.1060S (https://ui.adsabs.harvard.edu/abs/2017FoPh...47.1060S). doi:10.1007/s10701-017-0087-2 (https://doi.org/10.1007%2Fs10701-017-0087-2). S2CID 118438232 (https://api.semanticscholar.org/CorpusID:118438232).
23. Jin Wu, et al. "Hand-eye Calibration: 4D Procrustes Analysis Approach." (https://ieeexplore.ieee.org/document/8788685) 6 August 2019.
24. Wu, Jiasong, Ling Xu, Fuzhi Wu, Youyong Kong, Lotfi Senhadji, and Huazhong Shu. "Deep octonion networks." Neurocomputing (2020).

References

- Baez, John C. (2002). "The Octonions" (http://math.ucr.edu/home/baez/octonions/). *Bulletin of the American Mathematical Society*. **39** (2): 145–205. arXiv:math/0105155 (https://arxiv.org/abs/math/0105155). doi:10.1090/S0273-0979-01-00934-X (https://doi.org/10.1090%2FS0273-0979-01-00934-X). ISSN 0273-0979 (https://www.worldcat.org/issn/0273-0979). MR 1886087 (https://www.ams.org/mathscinet-getitem?mr=1886087). S2CID 586512 (https://api.semanticscholar.org/CorpusID:586512).
- Baez, John C. (2005). "Errata for *The Octonions*" (http://www.ams.org/journals/bull/2005-42-02/S0273-0979-05-01052-9/S0273-0979-05-01052-9.pdf) (PDF). *Bulletin of the American Mathematical Society*. **42** (2): 213–214. doi:10.1090/S0273-0979-05-01052-9 (https://doi.org/10.1090%2FS0273-0979-05-01052-9).
- Conway, John Horton; Smith, Derek A. (2003), *On Quaternions and Octonions: Their Geometry, Arithmetic, and Symmetry*, A. K. Peters, Ltd., ISBN 1-56881-134-9, Zbl 1098.17001 (https://zbmath.org/?format=complete&q=an:1098.17001). (Review (http://nugae.wordpress.com/2007/04/25/on-quaternions-and-octonions/)).
- Coxeter, H. S. M. (1946), "Integral Cayley numbers.", *Duke Math. J.*, **13** (4): 561–578, doi:10.1215/s0012-7094-46-01347-6 (https://doi.org/10.1215%2Fs0012-7094-46-01347-6), MR 0019111 (https://www.ams.org/mathscinet-getitem?mr=0019111)
- Dixon, Geoffrey M. (1994), *Division Algebras: Octonions, Quaternions, Complex Numbers and the Algebraic Design of Physics*, Kluvwer Academic Publishers, ISBN 0-7923-2890-6
- Freudenthal, Hans (1985) [1951], "Oktaven, Ausnahmegruppen und Oktavengeometrie", *Geom. Dedicata*, **19** (1): 7–63, doi:10.1007/BF00233101 (https://doi.org/10.1007%2FBF00233101), MR 0797151 (https://www.ams.org/mathscinet-getitem?mr=0797151), S2CID 121496094 (https://api.semanticscholar.org/CorpusID:121496094)

- Graves (1845), "On a Connection between the General Theory of Normal Couples and the Theory of Complete Quadratic Functions of Two Variables" (http://zs.thulb.uni-jena.de/receive/jportal_jparticle_00207304), *Phil. Mag.*, **26**: 315–320, doi:10.1080/14786444508645136 (https://doi.org/10.1080%2F14786444508645136)
- Kirmse (1924), "Über die Darstellbarkeit natürlicher ganzer Zahlen als Summen von acht Quadraten und über ein mit diesem Problem zusammenhängendes nichtkommutatives und nichtassoziatives Zahlensystem", *Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math. Phys. Kl.*, **76**: 63–82
- Lahti, Usko (2015), *Prof. Corvus Adamas: Luvut ja todistusmenetelmät. Johdanto matematiikan perusteisiin innokkaiden opiskelijoiden seurassa.*, Helsinki: Books on Demand, ISBN 978-952-318-558-6
- Salzmann, Helmut; Betten, Dieter; Grundhöfer, Theo; Hähl, Hermann; Löwen, Rainer; Stroppel, Markus (1995), *Compact Projective Planes, With an Introduction to Octonion Geometry*, De Gruyter Expositions in Mathematics, Walter de Gruyter, ISBN 3-11-011480-1, ISSN 0938-6572 (https://www.worldcat.org/issn/0938-6572), OCLC 748698685 (https://www.worldcat.org/oclc/748698685)
- van der Blij, F. (1961), "History of the octaves.", *Simon Stevin*, **34**: 106–125, MR 0130283 (https://www.ams.org/mathscinet-getitem?mr=0130283)

External links

- "Cayley numbers" (https://www.encyclopediaofmath.org/index.php?title=Cayley_numbers), *Encyclopedia of Mathematics*, EMS Press, 2001 [1994]
- Wilson, R. A. (2008), *Octonions* (http://www.maths.qmul.ac.uk/~raw/talks_files/octonions.pdf) (PDF), Pure Mathematics Seminar notes