GEOMETRY

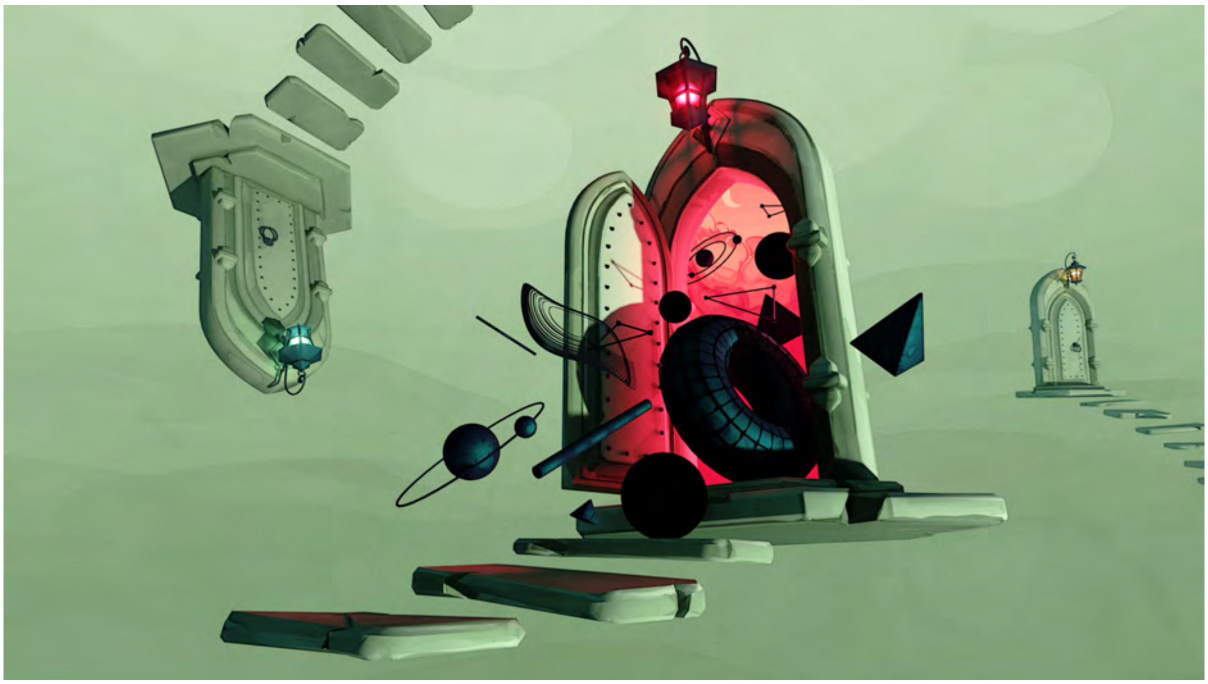
# Mathematicians Transcend Geometric Theory of Motion

By KELSEY HOUSTON-EDWARDS

*December 9, 2021*

*More than 30 years ago, Andreas Floer changed geometry. Now, two mathematicians have finally figured out how to extend his revolutionary perspective.*

8



Olena Shmahalo for Quanta Magazine

In a nearly 400-page paper posted in March, the mathematicians Mohammed Abouzaid and Andrew Blumberg of Columbia University have constructed a major extension of one of the biggest advances in geometry in recent decades. The work they built on relates to a well-known conjecture from the 1960s made by Vladimir Arnold. Arnold was studying classical mechanics and wanted to know when the orbits of planets are stable, returning to their original configuration after a set period.

Arnold's work was in an area of mathematics that concerns all the different configurations a physical system like bouncing billiard balls or orbiting planets can take. These configurations are encoded in geometric objects called phase spaces, which feature in a flourishing mathematical field called symplectic geometry.

Arnold predicted that every phase space of a certain type contains a minimum number of configurations in which the system it describes returns to where it started. This would be like billiard balls coming to occupy the same positions and velocities they held earlier. He anticipated that this minimum number is at least equal to the number of holes in the overall phase space, which can take the form of objects like a sphere (which has no holes) or a doughnut (which has one).

The Arnold conjecture linked two fundamentally different ways of thinking about a shape. It suggested that mathematicians could gain information about the motion of objects in a given shape (reflected in how many configurations return the object to where it started) in terms of its squishy topological properties (how many holes it has).

"Typically, symplectic things are harder than purely topological things. So being able to tell something symplectically from topological information is the main interest," said Ciprian Manolescu of Stanford University.

The first major advance on the Arnold conjecture took place decades later, in the 1980s, when a young mathematician named Andreas Floer developed a radical new way of counting holes. Floer's theory quickly became one of the central tools in symplectic geometry. Yet even as mathematicians used Floer's ideas, they imagined it should be possible to transcend his theory itself — to develop other theories in light of the new perspective that Floer opened up.

Finally, Abouzaid and Blumberg have done it. In their March paper they remake another important topological theory in terms of the techniques for counting holes that Floer pioneered. Echoing Floer's work, they then use this new theory to prove a version of the Arnold conjecture. This early proof-of-concept result has mathematicians anticipating that they'll eventually find many more uses for Abouzaid and Blumberg's ideas.

"It's a very important development for the field, both in terms of the theorem that it proves and the techniques it introduces," said Ailsa Keating of the University of Cambridge.

**The Geometry of Motion**

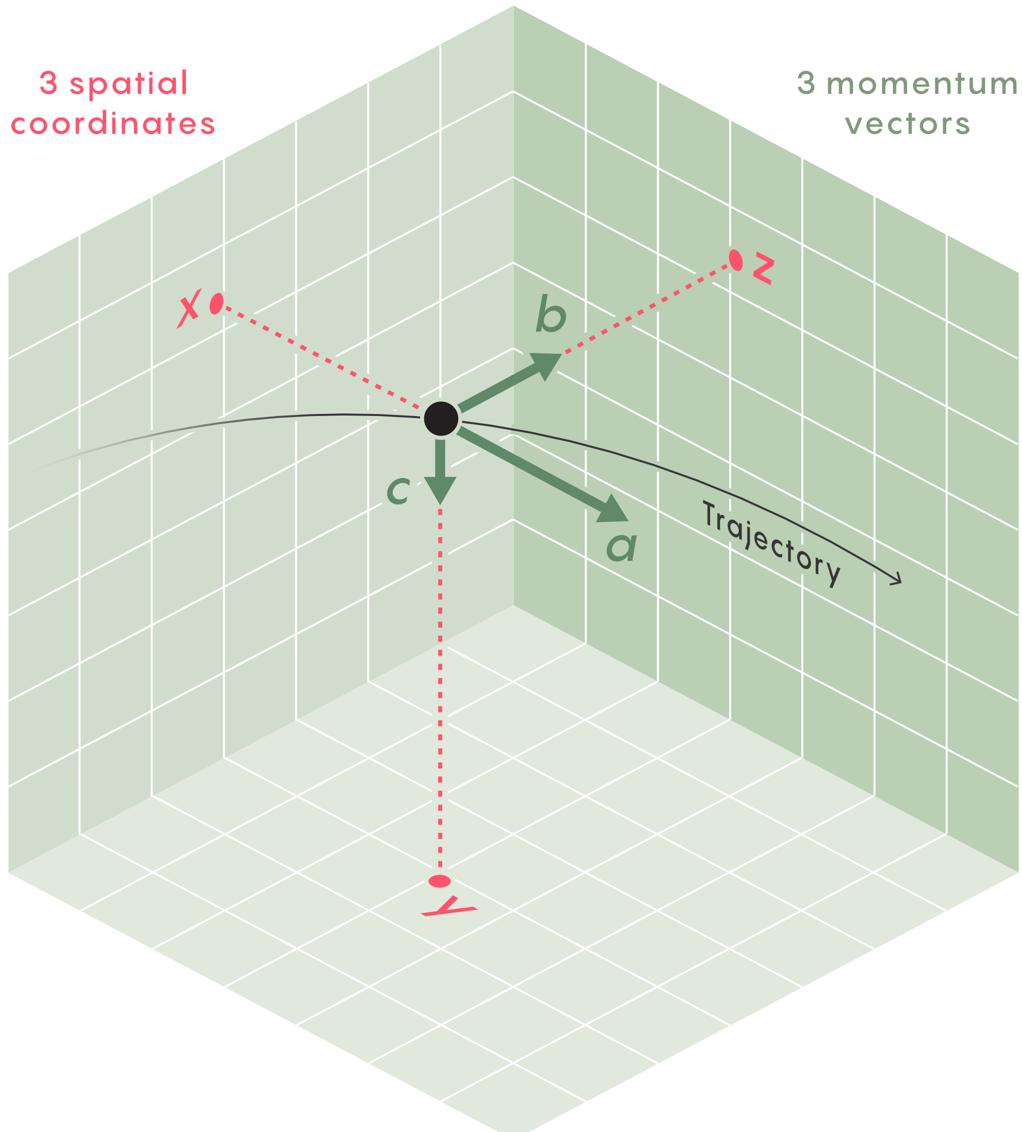
To get a sense for how configurations of a physical system can be used to build a geometric object, imagine a planet moving through space.

The planet's position and momentum can be described by six numbers, three for each property. If you represent each of the different configurations of the planet's position and momentum as a point with six coordinates, you'll create the phase space of the system. In this case, it has the shape of flat six-dimensional space. The motion of a single planet can be represented as a line weaving through this space.

Phase spaces can take on very different kinds of shapes. For example, the position of a swinging pendulum can be represented as a point on a circle and its momentum as a point on a line. The phase space of a pendulum is a circle crossed with a line, which forms a cylinder.

## Building a Phase Space

At each point in three-dimensional space you can assign three coordinates that specify a particle's position and three more coordinates that specify its momentum. Combine these six coordinates and you get a six-dimensional phase space.




—

Quanta Magazine

Symplectic geometry studies the properties of general phase spaces, called symplectic manifolds. On these manifolds, some paths loop back on themselves, forming closed orbits. Describing these closed orbits is a classic and challenging problem. Even a simpler question — does a physical system have any closed orbits? — is often difficult to answer.

That's why, in the 1960s, Vladimir Arnold sought to recast the tough task of counting closed orbits in terms of the simpler one of counting holes.

**Counting Holes**

Holes, like shapes, have different dimensions. One-dimensional holes resemble the inside of a rubber band. Two-dimensional holes occupy a region, like the inside of a balloon. Mathematicians study higher-dimensional holes, but they are nearly impossible to visualize.

Even in lower dimensions, our intuition about holes is shaky: Is a bowl a hole? How many holes does a straw have? In the field of topology, homology is the formal way to count holes. Homology associates to each shape an algebraic object, which can be used to extract information like the number of holes in each dimension.

To perform the association, mathematicians first break down the shape into component pieces that resemble triangles in different dimensions: one-dimensional lines, two-dimensional triangles, three-dimensional tetrahedra, and so on. Using a sort of algebra of shapes, topologists determine which components enclose a hole, the way three connected lines form a loop.

These computations are typically done using the integers, or whole numbers. But they can be done with other number systems, like the rational numbers (those which can be expressed as fractions) or cyclical number systems, which count in circles like a clock.

The various number systems produce different variants of the Arnold conjecture, since the question of relating the number of closed loops to the number of holes comes out a little differently depending on which number system you use to count those holes.

Abouzaid and Blumberg's recent paper proves the conjecture when the homology is computed with a cyclical number system. But to get there, they had to first build on the ideas of Andreas Floer, who, more than 30 years ago, created an entirely new theory that would eventually make it possible to compute the homology with rational numbers.

"Floer's work was obviously somehow revolutionary. Not just for this problem but for the way one would look at the field as a whole," said Ivan Smith of Cambridge.

**Floer's Perspective**

To prove the Arnold conjecture, Floer needed to count closed orbits. He started by drawing loops through the phase space and then combined neighboring loops to form geometric objects. He determined that the smallest of these geometric objects arose when the loops that formed them were closed orbits. These objects correspond to something called critical points.

Mathematicians already had a method, known as Morse theory, for studying these critical points. To understand Morse theory, imagine a torus suspended in a bucket which is slowly filling with water. The surface of the water changes shape at four different moments: when the rising water first touches the bottom of the torus, the bottom of the hole, the top of the hole and the top of the torus.

Samuel Velasco/Quanta Magazine

The rising water gives crucial topological information, which can be used to derive the shape's homology. In this way, Morse theory connects the critical points of a shape to its homology and therefore to the number of holes in each dimension.

"You kind of scan out the topology of the object," said Blumberg.

Morse theory was almost enough to solve the Arnold conjecture, but it has a limitation: It generally only works in finite dimensions. But Floer found a way to apply Morse theory to the infinite-dimensional spaces of loops he was interested in. His construction is known as Floer homology, and it became the bridge to solving the Arnold conjecture: The closed orbits in the Arnold conjecture become critical points in a space of loops, which are tied to the homology (or number of holes in the space) using Floer's modified version of Morse theory.

"[Floer] homology theory depends only on the topology of your manifold. [This] is Floer's incredible insight," said Agustin Moreno of the Institute for Advanced Study.

**Dividing by Zero**

Floer theory ended up being wildly useful in many areas of geometry and topology, including mirror symmetry and the study of knots.

"It's the central tool in the subject," said Manolescu.

But Floer theory did not completely resolve the Arnold conjecture because Floer's method only worked on one type of manifold. Over the next two decades, symplectic geometers engaged in a massive community effort to overcome this obstruction. Eventually, the work led to a proof of the Arnold conjecture where the homology is computed using rational numbers. But it didn't resolve the Arnold conjecture when holes are counted using other number systems, like cyclical numbers.

The reason the work didn't extend to cyclical number systems is that the proof involved dividing by the number of symmetries of a specific object. This is always possible with rational numbers. But with cyclical numbers, division is more finicky. If the number system cycles back after five — counting 0, 1, 2, 3, 4, 0, 1, 2, 3, 4 — then the numbers 5 and 10 are both equivalent to zero. (This is similar to the way 13:00 is the same as 1 p.m.) As a result, dividing by 5 in this setting is the same as dividing by zero — something forbidden in mathematics. It was clear that someone was going to have to develop new tools to circumvent this issue.

"If someone asked me what are the technical things that are preventing Floer theory from developing, the first thing that comes to mind is the fact that we have to introduce these denominators," said Abouzaid.

To expand Floer's theory and prove the Arnold conjecture with cyclical numbers, Abouzaid and Blumberg needed to look beyond homology.

**Climbing the Topologist's Tower**

Mathematicians often think of homology as the result of applying a specific recipe to a shape. During the 20th century, topologists began looking at homology on its own terms, independent of the process used to create it.

In the 1980s, Andreas Floer developed a radically new way of counting holes in topological shapes.

“Let’s not think about the recipe. Let’s think about what comes out of the recipe. What structure, what properties did this homology group have?” said Abouzaid.

Topologists sought out other theories that satisfied the same fundamental properties as homology. These became known as generalized homology theories. With homology at the base, topologists built up a tower of increasingly complicated generalized homology theories, all of which can be used to classify spaces.

Floer homology mirrors the ground-floor theory of homology. But symplectic geometers have long wondered if it's possible to develop Floer versions of topological theories higher up on the tower: theories that connect the generalized homology with specific features of a space in an infinite-dimensional setting, just as Floer's original theory did.

Floer never had a chance to attempt this work himself, dying in 1991 at the age of 34. But mathematicians continued to look for ways to expand his ideas.

**Benchmarking a New Theory**

Now, after nearly five years of work, Abouzaid and Blumberg have realized this vision. Their new paper develops a Floer version of Morava *K*-theory which they then use to prove the Arnold conjecture for cyclical number systems.

"There's a sense in which this completes a circle for us which ties all the way back to Floer's original work," said Keating.

Morava *K*-theory was created in the 1970s to expand the tower of topological theories. At the time it had no obvious connection to symplectic geometry or the Arnold conjecture. Like all general homology theories, Morava *K*-theory is an invariant, which means that it captures some essential and unchanging feature of an underlying shape. Abouzaid and Blumberg recognized that a Floer version of Morava *K*-theory was the key to proving a new version of the Arnold conjecture.

The original method failed to solve the Arnold conjecture with cyclical number systems because it involved dividing by a certain number of symmetries, a requirement that resulted from overcounting certain objects. But the Floer version of Morava *K*-theory doesn't require this division because each object is counted only once. As a result, mathematicians can now use it to count higher-dimensional holes and prove the Arnold conjecture using cyclical number systems.

But the authors are clear that their new invention — which is referred to as either Floer Morava *K*-theory or Floer homotopy theory — is not really about the Arnold conjecture.

"We didn't do this in order to solve the Arnold conjecture," said Blumberg. "The Arnold thing is like a sanity check to make sure you're doing the right kind of stuff."

Mathematicians are hopeful that the new Floer Morava *K*-theory will end up being useful for many problems, not just the Arnold conjecture. Abouzaid, with co-authors Smith and Mark McLean of Stony Brook University, has already put it to use in a new paper which answers a 25-year-old conjecture in symplectic geometry.

Other applications are almost certain to follow, and in ways that are hard to anticipate as mathematicians stand at the threshold of a new theory.

"That's one of the exciting things about math," said Jack Morava, a mathematician at Johns Hopkins University and the inventor of Morava *K*-theory. "You can go through a door and you wind up in a completely different universe. It's very much like *Alice in Wonderland*."

*Erica Klarreich contributed reporting for this article.*

***Correction:*** *December 10, 2021*
*The original version of this article stated that the phase space of a pendulum is a torus. It is actually a cylinder. The article has been updated accordingly.*