

The Amplituhedron

Nima Arkani-Hamed^a and Jaroslav Trnka^b

^a*School of Natural Sciences, Institute for Advanced Study,
1 Einstein Dr., Princeton, NJ 08540, U.S.A.*

^b*Walter Burke Institute for Theoretical Physics, California Institute of Technology,
1200 E Colorado Blvd., Pasadena, CA 91125, U.S.A.*

E-mail: arkani@ias.edu, trnka@caltech.edu

ABSTRACT: Perturbative scattering amplitudes in gauge theories have remarkable simplicity and hidden infinite dimensional symmetries that are completely obscured in the conventional formulation of field theory using Feynman diagrams. This suggests the existence of a new understanding for scattering amplitudes where locality and unitarity do not play a central role but are derived consequences from a different starting point. In this note we provide such an understanding for $\mathcal{N} = 4$ SYM scattering amplitudes in the planar limit, which we identify as “the volume” of a new mathematical object — the Amplituhedron — generalizing the positive Grassmannian. Locality and unitarity emerge hand-in-hand from positive geometry.

KEYWORDS: Supersymmetric gauge theory, Scattering Amplitudes

ARXIV EPRINT: [1312.2007](https://arxiv.org/abs/1312.2007)

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1 Scattering without space-time

Scattering amplitudes in gauge theories are amongst the most fundamental observables in physics. The textbook approach to computing these amplitudes in perturbation theory, using Feynman diagrams, makes locality and unitarity as manifest as possible, at the expense of introducing large amounts of gauge redundancy into our description of the physics, leading to an explosion of apparent complexity for the computation of amplitudes for all but the very simplest processes. Over the last quarter-century it has become clear that this complexity is a defect of the Feynman diagram approach to this physics, and is not present in the final amplitudes themselves, which are astonishingly simpler than indicated from the diagrammatic expansion [1–7].

This has been best understood for maximally supersymmetric gauge theories in the planar limit. Planar $\mathcal{N} = 4$ SYM has been used as a toy model for real physics in many

guises, but as toy models go, its application to scattering amplitudes is closer to the real world than any other. For instance the leading tree approximation to scattering amplitudes is identical to ordinary gluon scattering, and the most complicated part of loop amplitudes, involving virtual gluons, is also the same in $\mathcal{N} = 4$ SYM as in the real world.

Planar $\mathcal{N} = 4$ SYM amplitudes turn out to be especially simple and beautiful, enjoying the hidden symmetry of dual superconformal invariance [8, 9], associated with a dual interpretation of scattering amplitudes as a supersymmetric Wilson loop [10–12]. Dual superconformal symmetry combines with the ordinary conformal symmetry to generate an infinite dimension “Yangian” symmetry [13]. Feynman diagrams conceal this marvelous structure precisely as a consequence of making locality and unitarity manifest. For instance, the Yangian symmetry is obscured in either one of the standard physical descriptions either as a “scattering amplitude” in one space-time or a “Wilson-loop” in its dual.

This suggests that there must be a different formulation of the physics, where locality and unitarity do not play a central role, but emerge as derived features from a different starting point. A program to find a reformulation along these lines was initiated in [14, 15], and in the context of a planar $\mathcal{N} = 4$ SYM was pursued in [16–18], leading to a new physical and mathematical understanding of scattering amplitudes [19]. This picture builds on BCFW recursion relations for tree [6, 7] and loop [18, 20] amplitudes, and represents the amplitude as a sum over basic building blocks, which can be physically described as arising from gluing together the elementary three-particle amplitudes to build more complicated on-shell processes. These “on-shell diagrams” (which are essentially the same as the “twistor diagrams” of [16, 21, 22]) are remarkably connected with “cells” of a beautiful new structure in algebraic geometry, that has been studied by mathematicians over the past number of years, known as the positive Grassmannian [19, 23]. The on-shell building blocks can not be associated with local space-time processes. Instead, they enjoy all the symmetries of the theory, as made manifest by their connection with the Grassmannian — indeed, the infinite dimensional Yangian symmetry is easily seen to follow from “positive” diffeomorphisms [19].

While these developments give a complete understanding for the on-shell building blocks of the amplitude, they do not go further to explain *why* the building blocks have to be combined in a particular way to determine the full amplitude itself. Indeed, the particular combination of on-shell diagrams is dictated by *imposing* that the final result is local and unitary — locality and unitarity specify the singularity structure of the amplitude, and this information is *used* to determine the full integrand. This is unsatisfying, since we want to see locality and unitarity emerge from more primitive ideas, not merely use them to obtain the amplitude.

An important clue [17, 19, 24] pointing towards a deeper understanding is that the on-shell representation of scattering amplitudes is not unique: the recursion relations can be solved in many different ways, and so the final amplitude can be expressed as a sum of on-shell processes in different ways as well. The on-shell diagrams satisfy remarkable identities — now most easily understood from their association with cells of the positive Grassmannian — that can be used to establish these equivalences. This observation led Hodges [24] to a remarkable observation for the simplest case of “NMHV” tree amplitudes, further devel-

oped in [25]: the amplitude can be thought of as the volume of a certain polytope in momentum twistor space. However there was no a priori understanding of the origin of this polytope, and the picture resisted a direct generalization to more general trees or to loop amplitudes. Nonetheless, the polytope idea motivated a continuing search for a geometric representation of the amplitude as “the volume” of “some canonical region” in “some space”, somehow related to the positive Grassmannian, with different “triangulations” of the space corresponding to different natural decompositions of the amplitude into building blocks.

In this note we finally realize this picture. We will introduce a new mathematical object whose “volume” directly computes the scattering amplitude. We call this object the “Amplituhedron”, to denote its connection both to scattering amplitudes and positive geometry. The amplituhedron can be given a self-contained definition in a few lines as done below in section 9. We will motivate its definition from elementary considerations, and show how scattering amplitudes are extracted from it.

Everything flows from generalizing the notion of the “inside of a triangle in a plane”. The first obvious generalization is to the inside of a simplex in projective space, which further extends to the positive Grassmannian. The second generalization is to move from triangles to convex polygons, and then extend this into the Grassmannian. This gives us the amplituhedron for tree amplitudes, generalizing the positive Grassmannian by extending the notion of positivity to include external kinematical data. The full amplituhedron at all loop order further generalizes the notion of positivity in a way motivated by the natural idea of “hiding particles”.

Another familiar notion associated with triangles and polygons is their area. This is more naturally described in a projective way by a canonical 2-form with logarithmic singularities on the boundaries of the polygon. This form also generalizes to the full amplituhedron, and determines the (integrand of) the scattering amplitude. The geometry of the amplituhedron is completely bosonic, so the extraction of the superamplitude from this canonical form involves a novel treatment of supersymmetry, directly motivated by the Grassmannian structure.

The connection between the amplituhedron and scattering amplitudes is a conjecture which has passed a large number of non-trivial checks, including an understanding of how locality and unitarity arise as consequences of positivity. Our purpose in this note is to motivate and give the complete definition of the amplituhedron and its connection to the superamplitude in planar $\mathcal{N} = 4$ SYM. The discussion will be otherwise telegraphic and few details or examples will be given. In two accompanying notes [26, 27], we will initiate a systematic exploration of various aspects of the associated geometry and physics. A much more thorough exposition of these ideas, together with many examples worked out in detail, will be presented in [28].

Notation. The external data for massless n particle scattering amplitudes (for an excellent review see [29]) are labeled as $|\lambda_a, \tilde{\lambda}_a, \tilde{\eta}_a\rangle$ for $a = 1, \dots, n$. Here $\lambda_a, \tilde{\lambda}_a$ are the spinor-helicity variables, determining null momenta $p_a^{AA} = \lambda_a^A \tilde{\lambda}_a^{\dot{A}}$. The $\tilde{\eta}_a$ are (four) grassmann variables for on-shell superspace. The component of the color-stripped superamplitude

with weight $4(k+2)$ in the $\tilde{\eta}$'s is $M_{n,k}$. We can write

$$M_{n,k}(\lambda_a, \tilde{\lambda}_a, \tilde{\eta}_a) = \frac{\delta^4(\sum_a \lambda_a \tilde{\lambda}_a) \delta^8(\sum_a \lambda_a \tilde{\eta}_a)}{\langle 12 \rangle \dots \langle n1 \rangle} \times \mathcal{M}_{n,k}(z_a, \eta_a) \tag{1.1}$$

where (z_a, η_a) are the (super) ‘‘momentum-twistor’’ variables [24], with $z_a = \begin{pmatrix} \lambda_a \\ \mu_a \end{pmatrix}$. The z_a, η_a are unconstrained, and determine the $\lambda_a, \tilde{\lambda}_a$ as

$$\begin{aligned} \tilde{\lambda}_a &= \frac{\langle a-1 a \rangle \mu_{a+1} + \langle a+1 a-1 \rangle \mu_a + \langle a a+1 \rangle \mu_{a-1}}{\langle a-1 a \rangle \langle a a+1 \rangle}, \\ \tilde{\eta}_a &= \frac{\langle a-1 a \rangle \eta_{a+1} + \langle a+1 a-1 \rangle \eta_a + \langle a a+1 \rangle \eta_{a-1}}{\langle a-1 a \rangle \langle a a+1 \rangle} \end{aligned} \tag{1.2}$$

where throughout this paper, the angle brackets $\langle \dots \rangle$ denotes totally antisymmetric contraction with an ϵ tensor. $\mathcal{M}_{n,k}$ is cyclically invariant. It is also invariant under the little group action $(z_a, \eta_a) \rightarrow t_a(z_a, \eta_a)$, so (z_a, η_a) can be taken to live in $\mathbb{P}^{3|4}$.

At loop level, there is a well-defined notion of ‘‘the integrand’’ for scattering amplitudes, which at L loops is a $4L$ form. The loop integration variables are points in the (dual) spacetime x_i^μ , which in turn can be associated with L lines in momentum-twistor space that we denote as $\mathcal{L}_{(i)}$ for $i = 1, \dots, L$. The $4L$ form is [30–32]

$$\mathcal{M}(z_a, \eta_a; \mathcal{L}_{(i)}) \tag{1.3}$$

We can specify the line by giving two points $\mathcal{L}_{1(i)}, \mathcal{L}_{2(i)}$ on it, which we can collect as $\mathcal{L}_{\gamma(i)}$ for $\gamma = 1, 2$. \mathcal{L} can also be thought of as a 2 plane in 4 dimensions. In previous work, we have often referred to the two points on the line $\mathcal{L}_1, \mathcal{L}_2$ as ‘‘ AB ’’, and we will use this notation here as well.

Dual superconformal symmetry says that $\mathcal{M}_{n,k}$ is invariant under the $SL(4|4)$ symmetry acting on (z_a, η_a) as (super)linear transformations. The full symmetry of the theory is the Yangian of $SL(4|4)$.

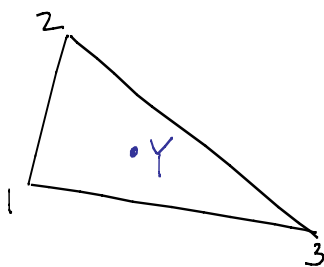
2 Triangles \rightarrow positive Grassmannian

To begin with, let us start with the simplest and most familiar geometric object of all, a triangle in two dimensions. Thinking projectively, the vertices are Z_1^I, Z_2^I, Z_3^I where $I = 1, \dots, 3$. The interior of the triangle is a collection of points of the form

$$Y^I = c_1 Z_1^I + c_2 Z_2^I + c_3 Z_3^I \tag{2.1}$$

where we span over all c_a with

$$c_a > 0 \tag{2.2}$$



More precisely, the interior of a triangle is associated with a triplet $(c_1, c_2, c_3)/GL(1)$, with all ratios $c_a/c_b > 0$, so that the c_a are either all positive or all negative, but here and in the generalizations that follow, we will abbreviate this by calling them all positive. Including the closure of the triangle replaces “positivity” with “non-negativity”, but we will continue to refer to this as “positivity” for brevity.

One obvious generalization of the triangle is to an $(n - 1)$ dimensional simplex in a general projective space, a collection $(c_1, \dots, c_n)/GL(1)$, with $c_a > 0$. The n -tuple $(c_1, \dots, c_n)/GL(1)$ specifies a line in n dimensions, or a point in \mathbb{P}^{n-1} . We can generalize this to the space of k -planes in n dimensions — the Grassmannian $G(k, n)$ — which we can take to be a collection of n k -dimensional vectors modulo $GL(k)$ transformations, grouped into a $k \times n$ matrix

$$C = \begin{pmatrix} & & & \\ & & & \\ c_1 & \dots & c_n & \end{pmatrix} / GL(k) \tag{2.3}$$

Projective space is the special case of $G(1, n)$. The notion of positivity giving us the “inside of a simplex” in projective space can be generalized to the Grassmannian. The only possible $GL(k)$ invariant notion of positivity for the matrix C requires us to fix a particular ordering of the columns, and demand that all minors in this ordering are positive:

$$\langle c_{a_1} \dots c_{a_k} \rangle > 0 \text{ for } a_1 < \dots < a_k \tag{2.4}$$

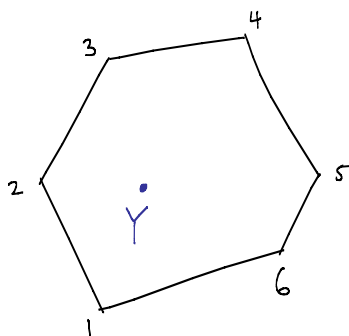
We can also talk about the very closely related space of positive matrices $M_+(k, n)$, which are just $k \times n$ matrices with all positive ordered minors. The only difference with the positive Grassmannian is that in $M_+(k, n)$ we are not modding out by $GL(k)$.

Note that while both $M_+(k, n)$ and $G_+(k, n)$ are defined with a given ordering for the columns of the matrices, they have a natural cyclic structure. Indeed, if (c_1, \dots, c_n) give a positive matrix, then positivity is preserved under the (twisted) cyclic action $c_1 \rightarrow c_2, \dots, c_n \rightarrow (-1)^{k-1}c_1$.

3 Polygons \rightarrow (tree) Amplituhedron $\mathcal{A}_{n,k}(Z)$

Another natural generalization of a triangle is to a more general polygon with n vertices Z_1^I, \dots, Z_n^I . Once again we would like to discuss the interior of this region. However in general this is not canonically defined — if the points Z_a are distributed randomly, they

don't obviously enclose a region where all the Z_a are all vertices. Only if the Z_a form a *convex* polygon do we have a canonical “interior” to talk about:



Now, convexity for the Z_a is a special case of positivity in the sense of the positive matrices we have just defined. The points Z_a form a closed polygon only if the $3 \times n$ matrix with columns Z_a has all positive (ordered) minors:

$$\langle Z_{a_1} Z_{a_2} Z_{a_3} \rangle > 0 \quad \text{for } a_1 < a_2 < a_3 \tag{3.1}$$

Having arranged for this, the interior of the polygon is given by points of the form

$$Y^I = c_1 Z_1^I + c_2 Z_2^I + \dots + c_n Z_n^I \quad \text{with } c_a > 0 \tag{3.2}$$

Note that this can be thought of as an interesting pairing of two different positive spaces. We have

$$(c_1, \dots, c_n) \in G_+(1, n), \quad (Z_1, \dots, Z_n) \in M_+(3, n) \tag{3.3}$$

If we jam them together to produce

$$Y^I = c_a Z_a^I \tag{3.4}$$

for fixed Z_a , ranging over all c_a gives us all the points on the inside of the polygon, living in $G(1, 3)$.

This object has a natural generalization to higher projective spaces; we can consider n points Z_a^I in $G(1, 1 + m)$, with $I = 1, \dots, 1 + m$, which are positive

$$\langle Z_{a_1} \dots Z_{a_{1+m}} \rangle > 0 \tag{3.5}$$

Then, the analog of the “inside of the polygon” are points of the form

$$Y^I = c_a Z_a^I, \quad \text{with } c_a > 0 \tag{3.6}$$

This space is very closely related to the “cyclic polytope” [33], which is the convex hull of n ordered points on the moment curve in \mathbb{P}^m , $Z_a = (1, t_a, t_a^2, \dots, t_a^m)$, with $t_1 < t_2 < \dots < t_n$. From our point of view, forcing the points to lie on the moment curve is overly restrictive; this is just one way of ensuring the positivity of the Z_a .

We can further generalize this structure into the Grassmannian. We take positive external data as $(k+m)$ dimensional vectors Z_a^I for $I = 1, \dots, k+m$. It is natural to restrict $n \geq (k+m)$, so that the external Z_a fill out the entire $(k+m)$ dimensional space. Consider the space of k -planes in this $(k+m)$ dimensional space, $Y \subset G(k, k+m)$, with co-ordinates

$$Y_\alpha^I, \alpha = 1, \dots, k, I = 1, \dots, k+m \tag{3.7}$$

We then consider a subspace of $G(k, k+m)$ determined by taking all possible “positive” linear combinations of the external data,

$$Y = C \cdot Z \tag{3.8}$$

or more explicitly

$$Y_\alpha^I = C_{\alpha a} Z_a^I \tag{3.9}$$

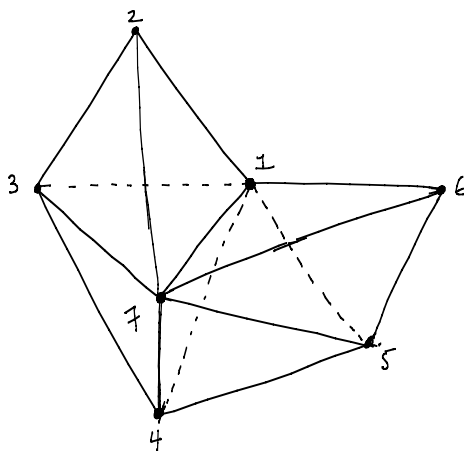
where

$$C_{\alpha a} \subset G_+(k, n), Z_a^I \subset M_+(k+m, n) \tag{3.10}$$

It is trivial to see that this space is cyclically invariant if m is even: under the twisted cyclic symmetry, $Z_n \rightarrow (-1)^{k+m-1} Z_1$ and $c_n \rightarrow (-1)^{k-1} c_1$, and the product is invariant for even m .

We call this space the generalized tree amplituhedron $\mathcal{A}_{n,k,m}(Z)$. The polygon is the simplest case with $k = 1, m = 2$. Another special case is $n = (k+m)$, where we can use $GL(k+m)$ transformations to set the external data to the identity matrix $Z_a^I = \delta_a^I$. In this case $\mathcal{A}_{k+m,k,m}$ is identical to the usual positive Grassmannian $G_+(k, k+m)$.

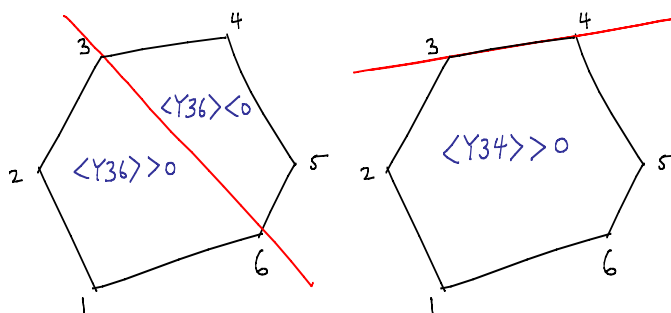
The case of immediate relevance to physics is $m = 4$, and we will refer to this as the tree amplituhedron $\mathcal{A}_{n,k}(Z)$. The tree amplituhedron lives in a $4k$ dimensional space and is not trivially visualizable. For $k = 1$, it is a polytope, with inequalities determined by linear equations, while for $k > 1$, it is not a polytope and is more “curvy”. Just to have a picture, below we sketch a 3-dimensional face of the 4 dimensional amplituhedron for $n = 8$, which turns out to be the space $Y = c_1 Z_1 + \dots c_7 Z_7$ for Z_a positive external data in \mathbb{P}^3 :



4 Why positivity?

We have motivated the structure of the amplituhedron by mimicking the geometric idea of the “inside” of a convex polygon. However there is a simpler and deeper origin of the need for positivity. We can attempt to define $Y = C \cdot Z$ with no positive restrictions on C or Z . But in general, this will not be projectively meaningful, and this expression won't allow us to define a region in $G(k, k+m)$. The reason is that for $n > k+m$, there is always some linear combination of the Z_a which sum to zero! We have to take care to avoid this happening, and in order to avoid “0” on the left hand side, we obviously need positivity properties on both the Z 's and the C 's.

It is simple and instructive to see why positivity ensures that the $Y = C \cdot Z$ map is projectively well-defined. We will see this as a by-product of locating the co-dimension one boundaries of the generalized tree amplituhedron. Let us illustrate the idea already for the simplest case of the polygon with $k = 1, m = 2$, with $Y = c_1 Z_1 + \dots c_n Z_n$. In order to look at the boundaries of the space, let us compute $\langle Y Z_i Z_j \rangle$ for some i, j . If as we sweep through all the allowed c 's, $\langle Y Z_i Z_j \rangle$ changes sign from being positive to negative, then somewhere $\langle Y Z_i Z_j \rangle \rightarrow 0$ and Y lies on the line $(Z_i Z_j)$ in the interior of the space, thus $(Z_i Z_j)$ should not be a boundary of the polygon. On the other hand, if $\langle Y Z_i Z_j \rangle$ everywhere has a uniform sign, then $(Z_i Z_j)$ is a boundary of the polygon:



Of course for the polygon it is trivial to directly see that the co-dimension one boundaries are just the lines $(Z_i Z_{i+1})$, but we wish to see this more algebraically, in a way that will generalize to the amplituhedron where “seeing” is harder. So, we compute

$$\langle Y Z_i Z_j \rangle = \sum_a c_a \langle Z_a Z_i Z_j \rangle \tag{4.1}$$

We can see why there is some hope for the positivity of this sum, since the $c_a > 0$, and also ordered minors of the Z 's are positive. It is however obvious that if i, j are not consecutive, some of the terms in this sum will be positive, but some (where a is stuck between i, j) will be negative. But precisely when i, j are consecutive, we get a manifestly positive sum:

$$\langle Y Z_i Z_{i+1} \rangle = \sum_a c_a \langle Z_a Z_i Z_{i+1} \rangle > 0 \tag{4.2}$$

Since $\langle Z_a Z_i Z_{i+1} \rangle > 0$ for $a \neq i, i+1$, this is manifestly positive. Thus the boundaries are lines $(Z_i Z_{i+1})$ as expected.

This also tells us that the map $Y = C \cdot Z$ is projectively well-defined. There is no way to get $Y \rightarrow 0$, since this would make the left hand side identically zero, which is impossible without making all the c_a vanish, which is not permitted as we mod out by $GL(1)$ on the c_a .

We can extend this logic to higher k, m . Let's look at the case $m = 4$ already for $k = 1$. We can investigate whether the plane $(Z_i Z_j Z_k Z_l)$ is a boundary by computing

$$\langle Y Z_i Z_j Z_k Z_l \rangle = \sum_a c_a \langle Z_a Z_i Z_j Z_k Z_l \rangle \tag{4.3}$$

Again, this is not in general positive. Only for (i, j, k, l) of the form $(i, i + 1, j, j + 1)$, we have

$$\langle Y Z_i Z_{i+1} Z_j Z_{j+1} \rangle = \sum_a c_a \langle Z_a Z_i Z_{i+1} Z_j Z_{j+1} \rangle > 0 \tag{4.4}$$

For general even the m , the boundaries are when Y is on the plane $(Z_i Z_{i+1} \dots Z_{i_{m/2-1}} Z_{i_{m/2}})$. This again shows that the $Y = C \cdot Z$ is projectively well-defined. The result extends trivially to general k , provided the positivity of C is respected. For $m = 4$ the boundaries are again when the k -plane $(Y_1 \dots Y_k)$ is on $(Z_i Z_{i+1} Z_j Z_{j+1})$, as follows from

$$\langle Y_1 \dots Y_k Z_i Z_{i+1} Z_j Z_{j+1} \rangle = \sum_{a_1 < \dots < a_k} \langle c_{a_1} \dots c_{a_k} \rangle \langle Z_{a_1} \dots Z_{a_k} Z_i Z_{i+1} Z_j Z_{j+1} \rangle > 0 \tag{4.5}$$

which also shows that Y is always a full rank k -plane in $k + 4$ dimensions.

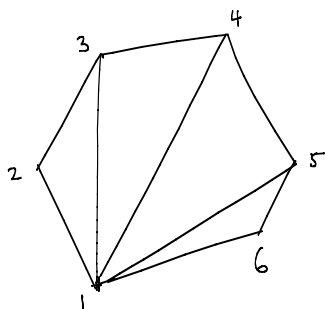
The emergence of boundaries on the plane $(Z_i Z_{i+1} Z_j Z_{j+1})$ is a simple and striking consequence of positivity. We will shortly understand that the location of these boundaries are the “positive origin” of locality from the geometry of the amplituhedron.

5 Cell decomposition

The tree amplituhedron can be thought of as the image of the top-cell of the the positive Grassmannian $G_+(k, n)$ under the map $Y = C \cdot Z$. Since $\dim G(k, k + m) = mk \leq \dim G(k, n) = k(n - k)$ for $n \geq k + m$, this is in general a highly redundant map. We can already see this in the simplest case of the polygon, which lives in 2 dimensions, while the c_a span an $(n - 1)$ dimensional space. The non-redundant maps into $G(k, k + m)$ can only come from the $m \times k$ dimensional “cells” of $G_+(k, n)$. For the polygon, these are the cells we can label as (i, j, k) , where all but (c_i, c_j, c_k) are non-vanishing. The image of these cells in the Y -space are of course just the triangles with vertices at Z_i, Z_j, Z_k , which lie inside the polygon.

The union of all these triangles covers the inside of the polygon. However, we can also cover the inside of the polygon more nicely with non-overlapping triangles, giving a triangulation. Said in a heavy-handed way, we find a collection of 2 dimensional cells of $G_+(1, n)$, so that their images in Y space are non-overlapping except on boundaries, and

collectively cover the entire polygon. Of course these collections of cells are not unique — there are many different triangulations of the polygon. A particularly simple one is



which we can write as

$$\sum_i (1\ i\ i + 1) \tag{5.1}$$

Sticking with $k = 1$ but moving to $m = 4$, the four-dimensional cells of $G_+(1, n)$ are labeled by five non-vanishing c 's (c_i, c_j, c_k, c_l, c_m). While it is harder to visualize, one can easily show algebraically that the above simple triangulation of the polygon generalizes to

$$\sum_{i < j} (1\ i\ i + 1\ j\ j + 1) \tag{5.2}$$

This expression is immediately recognizable to physicists familiar with scattering amplitudes in $\mathcal{N} = 4$ SYM. If the (i, j, k, l, m) are interpreted as “R-invariants”, this expression is one of the canonical BCFW representations of the $k = 1$ “NMHV” tree amplitudes. In the positive Grassmannian representation for amplitudes [17, 19], R-invariants are precisely associated with the corresponding four-dimensional cells of $G(1, n)$.

For general k, m any $m \times k$ dimensional cell of $G_+(k, n)$ maps under $Y = C \cdot Z$ into some region or cell in $G(k, k+m)$. Said more explicitly, consider an $m \times k$ dimensional cell Γ of the $G_+(k, n)$, with “positive co-ordinates” $C^\Gamma(\alpha_1^\Gamma, \dots, \alpha_{m \times k}^\Gamma)$ [19]. Putting $Y = C(\alpha) \cdot Z$ and scanning over all positive α 's, this carves out a region in $G(k, k+m)$ which is a corresponding cell Γ of the tree amplituhedron. A cell decomposition is a collection T of non-overlapping cells Γ which cover the entire amplituhedron.

The case of immediate relevance for physics is $m = 4$. For any k , the BCFW decomposition of tree amplitudes gives us a collection of $4 \times k$ dimensional cells of the positive Grassmannian. We have performed extensive checks for high k and n , that for positive external Z , under $Y = C \cdot Z$ these cells are beautifully mapped into non-overlapping regions of $G(k, k+4)$ that together cover the entire tree amplituhedron. As we have stressed, other than the desire to make the final result local and unitary, we did not previously have a rational for thinking about this particular collection of cells of $G_+(k, n)$. Now we know what natural question this collection of cells are answering: taken together they “cellulate” the tree amplituhedron. We will shortly see how to directly associate the amplitude itself directly with the geometry of the amplituhedron.

6 “Volume” as canonical form

Before discussing how to determine the (super)amplitude from the geometry, let us define the notion of a “volume” associated with the amplituhedron. As should by now be expected, we will merely generalize a simple existing idea from the world of triangles and polygons.

The usual notion of “area” has units and is obviously not projectively meaningful. However there is a closely related idea that is. For the triangle, we can consider a rational 2-form in Y -space, which has logarithmic singularities on the boundaries of the triangle. This is naturally associated with positive co-ordinates for the triangle, if we expand $Y = Z_3 + \alpha_1 Z_1 + \alpha_2 Z_2$, then the form is

$$\Omega_{123} = \frac{d\alpha_1 d\alpha_2}{\alpha_1 \alpha_2} \tag{6.1}$$

which can be re-written more invariantly as

$$\Omega_{123} = \frac{\langle Y dY dY \rangle \langle 123 \rangle^2}{\langle Y12 \rangle \langle Y23 \rangle \langle Y31 \rangle} \tag{6.2}$$

We can extend this to a form Ω_P for the convex polygon P . The defining property of Ω_P is that

Ω_P has logarithmic singularities on all the boundaries of P .

Ω_P can be obtained by first triangulating the polygon in some way, then summing the elementary two-form for each triangle, for instance as

$$\Omega_P = \sum_i \Omega_{1 i i+1}. \tag{6.3}$$

Each term in this sum has singularities corresponding to Y hitting the boundaries of the corresponding triangle. Most of these singularities, associated with the internal edges of the triangulation, are spurious, and cancel in the sum. Of course the full form Ω_P is independent of the particular triangulation.

This form is closely related to an area, not directly of the polygon P , but its dual \tilde{P} , which is also a convex polygon [25]. If we dualize so that points are mapped to lines and lines to points, then a point Y *inside* P is mapped to a line Y *outside* \tilde{P} . If we write $\Omega_P = \langle Y d^2 Y \rangle V(Y)$, then $V(Y)$ is the area of \tilde{P} living in the euclidean space defined by Y as the line at infinity.

This form can be generalized to the tree amplituhedron in an obvious way. We define a rational form $\Omega_{n,k}(Y; Z)$ with the property that

$\Omega_{n,k}(Y; Z)$ has logarithmic singularities on all the boundaries of $\mathcal{A}_{n,k}(Z)$.

Just as for the polygon, one concrete way of computing Ω is to give a cell decomposition of the amplituhedron. For any cell Γ associated with positive co-ordinate $(\alpha_1^\Gamma, \dots, \alpha_{4k}^\Gamma)$, there is an associated form with logarithmic singularities on the boundaries of the cell

$$\Omega_{n,k}^\Gamma(Y; Z) = \prod_{i=1}^{4k} \frac{d\alpha_i^\Gamma}{\alpha_i^\Gamma} \tag{6.4}$$

For instance, consider 4 dimensional cells for $k = 1$, associated with cells in $G_+(1, n)$ which are vanishing for all but columns a_1, \dots, a_5 , with positive co-ordinates $(\alpha_{a_1}, \dots, \alpha_{a_4}, \alpha_{a_5} = 1)$. Its image in Y space is simply

$$Y = \alpha_{a_1} Z_{a_1} + \dots + \alpha_{a_4} Z_{a_4} + Z_{a_5} \tag{6.5}$$

and the form is

$$\frac{d\alpha_{a_1}}{\alpha_{a_1}} \dots \frac{d\alpha_{a_4}}{\alpha_{a_4}} = \frac{\langle Y d^4 Y \rangle \langle Z_{a_1} Z_{a_2} Z_{a_3} Z_{a_4} Z_{a_5} \rangle^4}{\langle Y Z_{a_1} Z_{a_2} Z_{a_3} Z_{a_4} \rangle \dots \langle Y Z_{a_5} Z_{a_1} Z_{a_2} Z_{a_3} \rangle} \tag{6.6}$$

Now, given a collection of cells T that cover the full amplituhedron, $\Omega_{n,k}(Y; Z)$ is given by

$$\Omega_{n,k}(Y; Z) = \sum_{\Gamma \subset T} \Omega_{n,k}^\Gamma(Y; Z) \tag{6.7}$$

As with the polygon, the form is independent of the particular cell decomposition.

Note that the definition of the amplituhedron itself crucially depends on the positivity of the external data Z , and this geometry in turn determines the form Ω . However, once this form is in hand, it can be analytically continued to any (complex!) Y and Z .

7 The superamplitude

We have already defined central objects in our story: the tree amplituhedron, together with the associated form Ω that is loosely speaking its “volume”. The tree super-amplitude $\mathcal{M}_{n,k}$ can be directly extracted from $\Omega_{n,k}(Z)$. To see how, note that we can always use a $GL(4+k)$ transformation to send $Y \rightarrow Y_0$ as

$$Y_0 = \begin{pmatrix} 0_{4 \times k} \\ 1_{k \times k} \end{pmatrix} \tag{7.1}$$

We can think of the 4 dimensional space complementary to Y_0 , acted on by an unbroken $GL(4)$ symmetry, as the usual \mathbb{P}^3 of momentum-twistor space. Accordingly, we identify the top four components of the Z_a with the usual bosonic momentum-twistor variables z_a :

$$Z_a = \begin{pmatrix} z_a \\ *_{1} \\ \vdots \\ *_{k} \end{pmatrix} \tag{7.2}$$

We still have to decide how to interpret the remaining k entries of Z_a . Clearly, if they are normal bosonic variables, we have an infinite number of extra degrees of freedom. It is therefore natural to try and make the remaining components infinitesimal, by saying that some $\mathcal{N} + 1$ 'st power of them vanishes. This is equivalent to saying that each entry can be written as

$$Z_a = \begin{pmatrix} z_a \\ \phi_1^A \cdot \eta_{1A} \\ \vdots \\ \phi_k^A \cdot \eta_{Ak} \end{pmatrix} \tag{7.3}$$

where $\phi_{1,\dots,k}$ and η_a are Grassmann parameters, and $A = 1, \dots, \mathcal{N}$.

Now there is a unique way to extract the amplitude. We simply localize the form $\Omega_{n,k}(Y; Z)$ to Y_0 , and integrate over the ϕ 's:

$$\mathcal{M}_{n,k}(z_a, \eta_a) = \int d^{\mathcal{N}} \phi_1 \dots d^{\mathcal{N}} \phi_k \int \Omega_{n,k}(Y; Z) \delta^{4k}(Y; Y_0) \quad (7.4)$$

Here $\delta^{4k}(Y; Y_0)$ is a projective δ function

$$\delta^{4k}(Y; Y_0) = \int d^{k \times k} \rho_\alpha^\beta \det(\rho)^4 \delta^{k \times (k+4)}(Y_\alpha^I - \rho_\alpha^\beta Y_{0\beta}^I) \quad (7.5)$$

Note that there is really no integral to perform in the second step; the delta functions fully fix Y . Any form on $G(k, k+4)$ is of the form

$$\Omega = \langle Y_1 \dots Y_k d^4 Y_1 \rangle \dots \langle Y_1 \dots Y_k d^4 Y_k \rangle \times \omega_{n,k}(Y; Z) \quad (7.6)$$

and our expression just says that

$$\mathcal{M}_{n,k}(z_a, \eta_a) = \int d^{\mathcal{N}} \phi_1 \dots d^{\mathcal{N}} \phi_k \omega_{n,k}(Y_0; Z_a) \quad (7.7)$$

Note that we can define this operation for any \mathcal{N} , however, only for $\mathcal{N} = 4$ does $\mathcal{M}_{n,k}$ further have weight zero under the rescaling (z_a, η_a) .

This connection between the form and the super-amplitude also allows us to directly exhibit the relation between our super-amplitude expressions and the Grassmannian formulae of [17, 19]. Consider the form in Y -space associated with a given $4k$ dimensional cell Γ of $G_+(k, n)$. Then, if $C_{\alpha a}^\Gamma(\alpha_1, \dots, \alpha_{4k})$ are positive co-ordinates for the cell, and $\Omega^\Gamma = \frac{d\alpha_1^\Gamma}{\alpha_1^\Gamma} \dots \frac{d\alpha_{4k}^\Gamma}{\alpha_{4k}^\Gamma}$ is the associated form in Y space, then it is easy to show that

$$\int d^4 \phi_1 \dots d^4 \phi_k \int \Omega^\Gamma \delta^{4k}(Y; Y_0) = \int \frac{d\alpha_1^\Gamma}{\alpha_1^\Gamma} \dots \frac{d\alpha_{4k}^\Gamma}{\alpha_{4k}^\Gamma} \delta^{4k|4k}(C_{\alpha a}^\Gamma(z) Z_a) \quad (7.8)$$

where $Z_a = (z_a | \eta_a)$ are the super momentum-twistor variables. This is precisely the formula for computing on-shell diagrams (in momentum-twistor space) as described in [17, 19, 34]. Thus, while the amplituhedron geometry and the associated form Ω are purely bosonic, we have extracted from them super-amplitudes which are manifestly supersymmetric. Indeed, the connection to the Grassmannian shows much more — the superamplitude obtained for each cell is manifestly Yangian invariant [19].

8 Hiding particles \rightarrow loop positivity in $G_+(k, n; L)$

The direct generalization of “convex polygons” into the Grassmannian $G(k, k+4)$ has given us the tree amplituhedron. We will now ask a simple question: can we “hide particles” in a natural way? This will lead to an extended notion of positivity giving us loop amplitudes.

It is trivial to imagine what we might mean by hiding a single particle, but as we will see momentarily, the idea of hiding particles is only natural if we hide *pairs* of *adjacent*

particles. To pick a concrete example, suppose we have some positive matrix C with columns we'll label $(A_1, B_1, 1, 2, \dots, m, A_2, B_2, m+1, \dots, n)$. We can always gauge-fix the A_1, B_1 and A_2, B_2 columns so that the matrix takes the form

$$\begin{pmatrix} A_1 & B_1 & 1 & 2 & \dots & m & A_2 & B_2 & m+1 & \dots & n \\ \begin{pmatrix} 1 & 0 & * & * & \dots & * & 0 & 0 & * & \dots & * \\ 0 & 1 & * & * & \dots & * & 0 & 0 & * & \dots & * \\ 0 & 0 & * & * & \dots & * & 1 & 0 & * & \dots & * \\ 0 & 0 & * & * & \dots & * & 0 & 1 & * & \dots & * \\ 0 & 0 & * & * & \dots & * & 0 & 0 & * & \dots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & * & * & \dots & * & 0 & 0 & * & \dots & * \end{pmatrix} \end{pmatrix}$$

We would now like to “hide” the particles A_1, B_1, A_2, B_2 . We do this simply by chopping out the corresponding columns. The remaining matrix can be grouped into the form

$$\begin{pmatrix} -\frac{D_{(1)}}{C} - \\ -\frac{D_{(2)}}{C} - \\ C \end{pmatrix} \tag{8.1}$$

But the “hidden” particles leave an echo in the positivity properties of this matrix. The positivity of the minors involving all of (A_1, B_1, A_2, B_2) , (A_2, B_2) and (A_1, B_1) individually, as well those not involving A_1, B_1, A_2, B_2 at all enforce that the ordered maximal minors of the following matrices

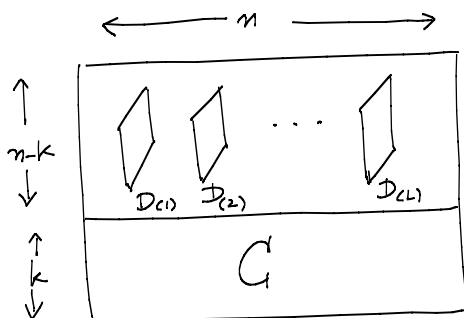
$$\begin{pmatrix} C \end{pmatrix}, \begin{pmatrix} -\frac{D_{(1)}}{C} - \\ C \end{pmatrix}, \begin{pmatrix} -\frac{D_{(2)}}{C} - \\ C \end{pmatrix}, \begin{pmatrix} -\frac{D_{(1)}}{C} - \\ -\frac{D_{(2)}}{C} - \\ C \end{pmatrix} \tag{8.2}$$

are all positive.

We can now see why particles are most naturally hidden in pairs. If we had instead hidden single particles as A_1, A_2, A_3, \dots in separate columns, the remaining minors would be positive or negative depending on the orderings of A_1, A_2, A_3, \dots , which is additional structure over and above the cyclic ordering of the external data. In order to avoid this arbitrariness, we should hide particles in even numbers, with pairs the minimal case. In order to ensure that only minors involving the pairs $(A_i B_i)$ are taken into account, we mod out by the $GL(2)$ action rotating the (A_i, B_i) columns into each other.

This “hidden particle” picture has thus motivated an extended notion of positivity associated with the Grassmannian. We are used to considering a k -plane in n dimensions C , with all ordered minors positive. But we can also imagine a collection of L 2-planes

$D_{(i)}$ in the $(n - k)$ dimensional complement of C , schematically



We call this space $G(k, n; L)$, and we will denote the collection of $(D_{(i)}, C)$ as \mathcal{C} . We can extend the notion of positivity to $G(k, n; L)$ by demanding that not only the ordered minors of C , but also of C with any collection of the $D_{(i)}$, are positive. (All minors must include the matrix C , since the $D_{(i)}$ are defined to live in the complement of C). Note that this notion is completely permutation invariant in the $D_{(i)}$.

Very interestingly, it turns out that while we motivated this notion of positivity by hiding particles from an underlying positive matrix, there are positive configurations of \mathcal{C} that can not be obtained by hiding particles from a positive matrix in this way.

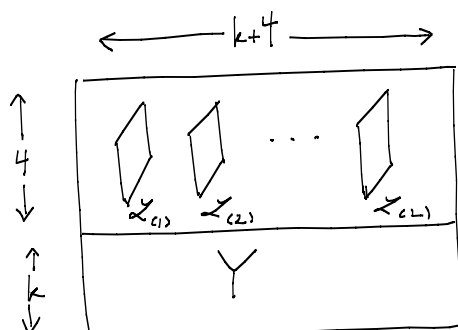
Extending the map $Y = C.Z$ in the obvious way to include the D 's leads us to define the full amplituhedron.

9 The Amplituhedron $\mathcal{A}_{n,k,L}(Z)$

We can now give the full definition of the amplituhedron $\mathcal{A}_{n,k,L}(Z)$. First, the external data for $n \geq k + 4$ particles is given by the vectors Z_a^I living in a $(4 + k)$ dimensional space; where $a = 1, \dots, n$ and $I = 1, \dots, 4 + k$. The data is positive

$$\langle Z_{a_1} \dots Z_{a_{4+k}} \rangle > 0 \quad \text{for } a_1 < \dots < a_{4+k} \tag{9.1}$$

The amplituhedron lives in $G(k, k + 4; L)$: the space of k planes Y in $(k + 4)$ dimensions, together with L 2-planes $\mathcal{L}_{(i)}$ in the 4 dimensional complement of Y , schematically



We will denote the collection of $(\mathcal{L}_{(i)}, Y)$ as \mathcal{Y} .

The amplituhedron $\mathcal{A}_{n,k,L}(Z)$ is the subspace of $G(k, k+4; L)$ consisting of all \mathcal{Y} 's which are a positive linear combination of the external data,

$$\mathcal{Y} = \mathcal{C} \cdot Z \tag{9.2}$$

More explicitly in components, the k -plane is Y_α^I , and the 2-planes are $\mathcal{L}_{\gamma(i)}^I$, where $\gamma = 1, 2$ and $i = 1, \dots, L$. The amplituhedron is the space of all $Y, \mathcal{L}_{(i)}$ of the form

$$Y_\alpha^I = C_{\alpha a} Z_a^I, \quad \mathcal{L}_{\gamma(i)}^I = D_{\gamma a(i)} Z_a^I \tag{9.3}$$

where as in the previous section the $C_{\alpha a}$ specifies a k -plane in n -dimensions, and the $D_{\gamma a(i)}$ are L 2-planes living in the $(n-k)$ dimensional complement of C , with the positivity property that for any $0 \leq l \leq L$, all the ordered $(k+2l) \times (k+2l)$ minors of the $(k+2l) \times n$ matrix

$$\begin{pmatrix} - & D_{(i_1)} & - \\ & \vdots & \\ - & D_{(i_l)} & - \\ - & C & - \end{pmatrix} \tag{9.4}$$

are positive.

The notion of cells, cell decomposition and canonical form can be extended to the full amplituhedron. A cell Γ is associated with a set of positive co-ordinates $\alpha^\Gamma = (\alpha_1^\Gamma, \dots, \alpha_{4(k+L)}^\Gamma)$, rational in the \mathcal{C} , such that for α 's positive, $\mathcal{C}(\alpha) = (D_{(i)}(\alpha), C(\alpha))$ is in $G_+(k, n; L)$. A cell decomposition is a collection T of non-intersecting cells Γ whose images under $\mathcal{Y} = \mathcal{C} \cdot Z$ cover the entire amplituhedron. The rational form $\Omega_{n,k,L}(\mathcal{Y}; Z)$ is defined by having the property that

$$\Omega_{n,k,L}(Y; Z) \text{ has logarithmic singularities on all the boundaries of } \mathcal{A}_{n,k,L}(Z)$$

A concrete formula follows from a cell decomposition as

$$\Omega_{n,k,L}(\mathcal{Y}; Z) = \sum_{\Gamma \in T} \prod_{i=1}^{4(k+L)} \frac{d\alpha_i^\Gamma}{\alpha_i^\Gamma} \tag{9.5}$$

Of course any cell decomposition gives the same form $\Omega_{n,k,L}$.

10 The loop amplitude form

We can extract the $4L$ -form for the loop integrand from $\Omega_{n,k,L}$ in the obvious way. The 2-planes $\mathcal{L}_{(i)}$, being in the complement of Y_0 , can be taken to be non-vanishing in the first 4 entries $\mathcal{L}_{(i)}^I = (\mathcal{L}_{(i)2 \times 4} | 0_{2 \times k})$. Each $\mathcal{L}_{\gamma(i)}$ gives us a line $(\mathcal{L}_{\gamma=1} \mathcal{L}_{\gamma=2})_{(i)}$ (which we have also been calling $(AB)_{(i)}$ in \mathbb{P}^3). These are the momentum-twistor representation of the loop integration variables. The analog of equation (7.4) for the loop integrand form is

$$\mathcal{M}_{n,k}(z_a, \eta_a; \mathcal{L}_{(\gamma(i))}) = \int d^4 \phi_1 \dots d^4 \phi_k \int \Omega_{n,k,L}(Y, \mathcal{L}_{\gamma(i)}; Z) \delta^{4k}(Y; Y_0) \tag{10.1}$$

Any form on $G(k, k + 4k; L)$ can be written as

$$\Omega = \langle Y d^4 Y_1 \rangle \dots \langle Y d^4 Y_k \rangle \prod_{i=1}^L \langle Y \mathcal{L}_{1(i)} \mathcal{L}_{2(i)} d^2 \mathcal{L}_{1(i)} \rangle \langle Y \mathcal{L}_{1(i)} \mathcal{L}_{2(i)} d^2 \mathcal{L}_{2(i)} \rangle \times \omega_{n,k,L}(Y, \mathcal{L}_{(i)})(Z) \quad (10.2)$$

where we denoted $Y = Y_1 \dots Y_k$. So we have for the integrand of the all-loop amplitude

$$\mathcal{M}_{n,k}(z_a, \eta_a, \mathcal{L}_{\gamma(i)}) = \int d^4 \phi_1 \dots d^4 \phi_k \prod_{i=1}^L \langle \mathcal{L}_{1(i)} \mathcal{L}_{2(i)} d^2 \mathcal{L}_{1(i)} \rangle \langle \mathcal{L}_{1(i)} \mathcal{L}_{2(i)} d^2 \mathcal{L}_{2(i)} \rangle \omega_{n,k}(Y_0, \mathcal{L}_{\gamma(i)}; Z_a) \quad (10.3)$$

Already the simplest case $k = 0$ of the amplituhedron is interesting at loop level. At 1-loop, we have a 2-plane in 4 dimensions AB , and the D matrix is just restricted to be in $G_+(2, n)$. It is easy to see that the 4 dimensional cells of $G_+(2, n)$ are labeled by a pair of triples $[a, b, c; x, y, z]$, where the top row of the matrix is non-zero in the columns (a, b, c) and the bottom in columns (x, y, z) . A simple collection of these

$$\sum_{i < j} [1 \ i \ i + 1; 1 \ j \ j + 1] \quad (10.4)$$

beautifully covers the amplituhedron in this case. The map into $G(2, 4)$ for each cell is

$$A = Z_1 + \alpha_i Z_i + \alpha_{i+1} Z_{i+1}, \quad B = -Z_1 + \alpha_j Z_j + \alpha_{j+1} Z_{j+1} \quad (10.5)$$

and so the form associated with the cell is

$$\frac{d\alpha_i \ d\alpha_{i+1} \ \alpha_j \ d\alpha_{j+1}}{\alpha_i \ \alpha_{i+1} \ \alpha_j \ \alpha_{j+1}} = \frac{\langle AB d^2 A \rangle \langle AB d^2 B \rangle \langle AB(1 \ i \ i + 1) \cap (1 \ j \ j + 1) \rangle^2}{\langle AB \ 1 \ i \rangle \langle AB \ 1 \ i + 1 \rangle \langle AB \ i \ i + 1 \rangle \langle AB \ 1 \ j \rangle \langle AB \ 1 \ j + 1 \rangle \langle AB \ j \ j + 1 \rangle} \quad (10.6)$$

The form Ω gives exactly the “Kermit” expansion for the MHV integrand given in [18], now obtained without any reference to tree amplitudes, forward limits or recursion relations.

In this simple case, direct triangulation of the space is straightforward. But we could also have worked backwards, starting with the BCFW formula, and recognizing how each term in the “Kermit” expansion is associated with positive co-ordinates for some cell of the amplituhedron. We could then observe that, remarkably, these cells are non-overlapping, and together cover the full amplituhedron.

In order to illustrate more of the structure of the loop amplituhedron, including the interplay between the “ C ” and “ D ” matrices, let us consider the 1-loop $k = 1$ amplitude for $n = 6$. There are 16 terms in the BCFW recursion, which can all be mapped back to their Y, AB space form, and in turn associated with positive co-ordinates in the amplituhedron. For instance, one of BCFW terms is

$$\frac{\langle Y AB 13 \rangle \langle Y AB(561) \cap (2345) \rangle^4 \langle Y AB(123) \cap (Y456) \rangle^2}{\langle Y 2345 \rangle \langle Y AB(561) \cap (Y345) \rangle \langle Y AB(561) \cap (Y234) \rangle \langle Y AB(561) \cap (Y235) \rangle \langle Y AB 56 \rangle \langle Y AB(561) \cap (Y45(23) \cap (YAB1)) \rangle \langle Y AB 12 \rangle \langle Y AB 23 \rangle \langle Y AB 13 \rangle \langle Y AB 15 \rangle \langle Y AB 16 \rangle}$$

While it may not be immediately apparent, this is nothing but the “dlog” canonical form associated with the following positive co-ordinates for the (D, C) matrix

$$\begin{pmatrix} D \\ C \end{pmatrix} = \begin{pmatrix} 1 & x & y & 0 & 0 & 0 \\ -w & 0 & 0 & 0 & -1 & -z \\ w & xt_1 & t_2 + t_1y & t_3 & 1 + t_4 & z \end{pmatrix}$$

This exercise can be repeated with all 16 BCFW terms. The corresponding (D, C) matrices are

$$\begin{aligned} & \begin{pmatrix} 0 & 0 & 1 & x & y & 0 \\ 0 & -t_1 & -t_2 - w & 0 & zt_3 & -zt_4 \\ -1 & t_1 & t_2 & 0 & t_3 & t_4 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & x & y & 0 \\ 0 & 0 & -w & 0 & z + t_3 & -t_4 \\ -1 & t_1 & t_2 & 0 & t_3 & t_4 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 & x + t_3y & yt_4 \\ 0 & -t_1w & -z - t_2w & 0 & t_3 & -t_4 \\ -1 & t_1 & t_2 & 0 & t_3 & t_4 \end{pmatrix} \\ & \begin{pmatrix} 0 & 0 & t_1 & t_2 + x + yw & y & 0 \\ 0 & 0 & 0 & wz & z + t_3 & -t_4 \\ -1 & 0 & t_1 & t_2 & t_3 & t_4 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & x & y & 0 \\ w & 0 & 0 & 0 & 1 & z \\ -1 & t_1 & t_2 & t_3 & t_4 + x & y \end{pmatrix} \begin{pmatrix} 1 & 0 & -x & -w & 0 & 0 \\ -1 & 0 & 0 & 0 & y + wt_4 & -z \\ -1 & t_1 & t_2 & t_3 + xt_4 & y & 0 \end{pmatrix} \\ & \begin{pmatrix} 1 & x & y & 0 & 0 & 0 \\ -w & 0 & 0 & 0 & -1 & -z \\ w & xt_1 & t_2 + t_1y & t_3 & 1 + t_4 & z \end{pmatrix} \begin{pmatrix} 0 & t_2 & t_3 + x + yw & t_4 + z & 0 & 0 \\ 0 & 0 & zw & t_4 + z & 1 & 0 \\ -1 & t_1 & t_2 & t_3 & t_4 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & x & y & 0 \\ w & 0 & 0 & 0 & 0 & 1 \\ -1 & t_1 & t_2 & t_3 + xt_4 & y & 0 \end{pmatrix} \\ & \begin{pmatrix} -1 & 0 & 0 & x & y & 0 \\ w & 0 & 0 & 0 & 1 & z \\ -1 & t_1 + w & 0 & t_2 & t_3 + xt_4 & 1 + t_4y \end{pmatrix} \begin{pmatrix} x & 1 & y & 0 & 0 & 0 \\ 0 & w & -z + wy + t_1 & t_2 & 0 & -t_4 \\ -1 & t_1 & t_2 & t_3 & t_4 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & x & y & 0 \\ w & 0 & 0 & 0 & 0 & 1 \\ -1 & t_1 & t_2 & t_3 & t_4 + x & y \end{pmatrix} \\ & \begin{pmatrix} 0 & x & 1 & w & 0 & 0 \\ -z & -y & 0 & 0 & -1 & 0 \\ -z & z(1+t_1) & -y & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & -x & -y & 0 & 0 \\ w & 0 & 0 & 0 & 1 & z \\ -1 & t_1 & t_2 & t_3 + t_1y & 1 + t_3 + t_4 & z \end{pmatrix} \\ & \begin{pmatrix} -1 & -x & -y & 0 & 0 & 0 \\ w & 0 & 0 & 0 & 1 & z \\ -1 & t_1 & t_2 & 1 + t_3 + t_4 & z & z(1+t_4) \end{pmatrix} \begin{pmatrix} -1 & -x & -y & 0 & 0 & 0 \\ w & t_1x & t_2 + yt_1 & 0 & 1 & z \\ -1 & t_1 & t_2 & 1 + t_3 + t_4 & z & z(1+t_4) \end{pmatrix} \end{aligned}$$

One can easily check that for all variables positive, the bottom row of these matrices is positive, and all the ordered 3×3 minors are also positive. For any cell, we can range over all the positive variables, which under the $\mathcal{Y} = \mathcal{C} \cdot \mathcal{Z}$ gives an image of the cell in (Y, AB) space. Remarkably, we find that these cells are non-overlapping, and cover the entire space. This can be checked directly in a simple way. We begin by fixing positive external data (Z_1, \dots, Z_6) . We then choose any positive matrix \mathcal{C} at random, which gives an associated point \mathcal{Y} inside the amplituhedron. We can ask whether or not this point is contained in one of the cells, by seeing whether \mathcal{Y} can be reproduced with positive values for all eight variables of the cell. Doing this we find that every point in the amplituhedron is contained in just one of these cells (except of course for points on the common boundaries of different cells). The cells taken together therefore give a cellulation of the amplituhedron.

Note that the form shown above, associated with a BCFW term, has some physical poles (like $\langle YAB12 \rangle$), but also many unphysical poles. The unphysical poles are associated with boundaries of the cell that are “inside” the amplituhedron, and not boundaries of the amplituhedron themselves. These boundaries are spurious, and so are the corresponding poles, which cancel in the sum over all BCFW terms.

We have checked in many other examples, for higher k and also at higher loops, that (a) BCFW terms can be expressed as canonical forms associated with cells of the amplituhedron and (b) these collection of cells do cover the amplituhedron.

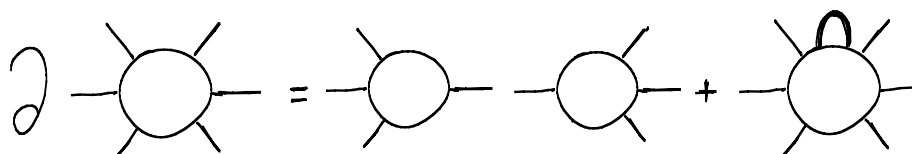
It is satisfying to have a definition of the loop amplituhedron that lives directly in the space relevant for loop amplitudes. This is in contrast with the approach to computing the loop integrand using recursion relations, which ultimately traces back to higher k and

n tree amplitudes. Consider the simple case of the 2-loop 4-particle amplitude. We are after a form in the space of two 2-planes $(AB)_1, (AB)_2$ in four dimensions. The BCFW approach begins with the $k = 2, n = 8$ tree amplitudes, and arrives at the form we are interested in after taking two “forward limits”. But the amplituhedron lives directly in the $(AB)_1, (AB)_2$ space, and we can find a cell decomposition for it directly, yielding the form without having to refer to any tree amplitudes.

We have understood how to directly “cellulate” the amplituhedron in a number of other examples, and strongly suspect that there will be a general understanding for how to do this. The BCFW decomposition of tree amplitudes seems to be associated with particularly nice, canonical cellulations of the tree amplituhedron. Loop level BCFW also gives a cell decomposition. The “direct” cellulations we have found in many cases are however simpler, without an obvious connection to the BCFW expansion.

11 Locality and unitarity from positivity

Locality and unitarity are encoded in the positive geometry of the amplituhedron in a beautiful way. As is well-known, locality and unitarity are directly reflected in the singularity structure of the integrand for scattering amplitudes. In momentum-twistor language, the only allowed singularities at tree-level should occur when $\langle Z_i Z_{i+1} Z_j Z_{j+1} \rangle \rightarrow 0$; in the loop-level integrand, we can also have poles of the form $\langle AB i i + 1 \rangle \rightarrow 0$, and $\langle AB_{(i)} AB_{(j)} \rangle \rightarrow 0$. Unitarity is reflected in what happens as poles are approached, schematically we have [19]



Given the connection between the form $\Omega_{n,k,L}$ and the amplitude, it is obvious that the first (co-dimension one) poles of the amplitude are associated with the co-dimension one “faces” of the amplituhedron. For trees, we have already seen that, remarkably, positivity forces these faces to be precisely where $\langle Y_1 \dots Y_k Z_i Z_{i+1} Z_j Z_{j+1} \rangle \rightarrow 0$, exactly as needed for locality. The analog statement for the full loop amplituhedron also obviously includes $\langle Y_1 \dots Y_k AB i i + 1 \rangle \rightarrow 0$.

The factorization properties of the amplitude also follow directly as a consequence of positivity. For instance, let us consider the boundary of the tree amplituhedron where the k plane $(Y_1 \dots Y_k)$ is on the plane $(Z_i Z_{i+1} Z_j Z_{j+1})$. We can e.g. assume that Y_1 is a linear combination of $(Z_i, Z_{i+1}, Z_j, Z_{j+1})$, and thus that the top row of the C matrix is only non-zero in these columns. But then, positivity remarkably forces the C matrix to

“factorize” in the form

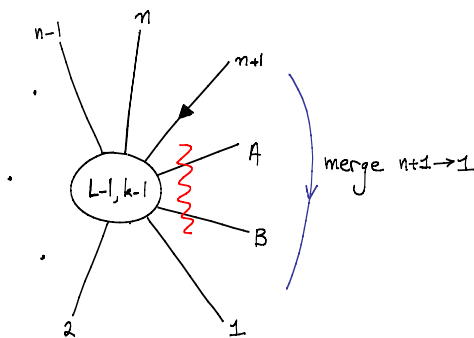
$$\begin{array}{c}
 \uparrow \\
 k_L \\
 \downarrow
 \end{array}
 \begin{array}{cccccc}
 & i & i+1 & & j & j+1 \\
 \left(\begin{array}{cccccc}
 * & * & 0 & \dots & 0 & * & * & 0 & \dots & 0 \\
 & & & & & & & & & & 0 & \dots & 0 \\
 & & & & & & & & & & 0 & \dots & 0 \\
 & & & & & & & & & & 0 & \dots & 0 \\
 & & & & & & & & & & & & & \uparrow \\
 & & & & & & & & & & & & & k_R \\
 & & & & & & & & & & & & & \downarrow
 \end{array} \right)
 \end{array}$$

for all possible k_L, k_R such that $k_L + k_R = k - 1$. This factorized form of the C matrix in turn implies that on this boundary, the amplituhedron does “split” into lower-dimensional amplituhedra in exactly the way required for the factorization of the amplitude.

We can similarly understand the single-cut of the loop integrand. Consider for concreteness the simplest case of the n particle one-loop MHV amplitude. On the boundary where $\langle ABn1 \rangle \rightarrow 0$, the D matrix has the form

$$\begin{array}{cccc}
 & 1 & 2 & \dots & n \\
 \left(\begin{array}{cccc}
 1 & 0 & \dots & -x_n \\
 y_1 & y_2 & \dots & y_n
 \end{array} \right)
 \end{array}$$

The connection of this D matrix to the forward limit [35] of the NMHV tree amplitude is simple. In the language of [18], the forward limit in momentum-twistor space is represented as



we start with the tree NMHV amplitude, associated with the positive $1 \times n$ matrix

$$(y_A y_B y_1 y_2 \dots y_n) \tag{11.1}$$

and first we “add” particle $n + 1$ between n and A , which adds three degrees of freedom x_n, x_A, α

$$\begin{array}{cccccc}
 & A & B & 1 & 2 & \dots & n & n+1 \\
 \left(\begin{array}{cccccc}
 x_A & \alpha x_A & 0 & 0 & \dots & -x_n & -1 \\
 y_A & y_B + \alpha y_A & y_1 & y_2 & \dots & y_n & 0
 \end{array} \right)
 \end{array}$$

and we finally “merge” $n + 1, 1$, which means shifting column 1 as $c_1 \rightarrow c_1 - c_{n+1}$ and removing column $(n + 1)$. This gives us the matrix

$$\begin{pmatrix} A & B & 1 & 2 & \dots & n \\ x_A & \alpha x_A & 1 & 0 & \dots & -x_n \\ y_A & y_B + \alpha y_A & y_1 & y_2 & \dots & y_n \end{pmatrix}$$

note that the the A, B columns have precisely four degrees of freedom x_A, α, y_A, y_B which we can remove by $GL(2)$ acting on the A, B columns. Chopping off A, B we are then left precisely with the D matrix on the single cut. This shows that the single cut of the loop integrand is the forward limit of the tree amplitude, exactly as required by unitarity.

12 Four particles at all loops

Let us briefly describe the simplest example illustrating the novelties of positivity at loop level, for four-particle scattering at L loops. We can parametrize each $D_{(i)}$ as

$$D_{(i)} = \begin{pmatrix} 1 & x_i & 0 & -w_i \\ 0 & y_i & 1 & z_i \end{pmatrix} \tag{12.1}$$

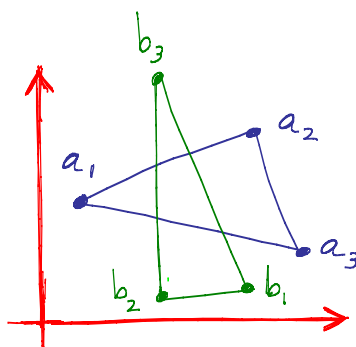
In this simple case the positivity constraints are just that all the 2×2 minors of $D_{(i)}$ and the 4×4 minors

$$\det \begin{pmatrix} - & D_{(i)} & - \\ - & D_{(j)} & - \end{pmatrix} \tag{12.2}$$

are positive. This translates to

$$x_i, y_i, z_i, w_i > 0, \quad (x_i - x_j)(z_i - z_j) + (y_i - y_j)(w_i - w_j) < 0 \tag{12.3}$$

We can rephrase this problem in a simple, purely geometrical way by defining two dimensional vectors $\vec{a}_i = (x_i, y_i), \vec{b}_i = (z_i, w_i)$. The points are in the upper quadrant of the plane. The mutual positivity condition is just $(\vec{a}_i - \vec{a}_j) \cdot (\vec{b}_i - \vec{b}_j) < 0$. Geometrically this just means that the \vec{a}, \vec{b} must be arranged so that for every pair i, j , the line directed from $\vec{a}_i \rightarrow \vec{a}_j$ is pointed in the opposite direction as the one directed from $\vec{b}_i \rightarrow \vec{b}_j$. An example of an allowed configuration of such points for $L = 3$ is



Finding a cell decomposition of this $4L$ dimensional space directly gives us the integrand for the four-particle amplitude at L -loops.

Now, we know that the final form can be expressed as a sum over local, planar diagrams. This makes it all the more remarkable that no-where in the definition of our geometry problem do we reference to diagrams of any sort, planar or not! Nonetheless, this property is one of many that emerges from positivity.

As we will describe at greater length in [26], it is easy to find a cell decomposition for the full space “manually” at low-loop orders. We suspect there is a more systematic approach to understanding the geometry that might crack the problem at all loop order. As an interesting warmup to the full problem, we can investigate lower-dimensional “faces” of the four-particle amplituhedron. Cellulations of these faces corresponds to computing certain cuts of the integrand, at all loop orders. We will discuss many of these faces and cuts systematically in [26]. Here we will content ourselves by presenting some especially simple but not completely trivial examples.

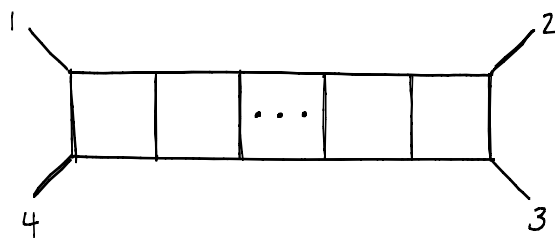
Let us start by considering an extremely simple boundary of the space, where all the $w_i \rightarrow 0$. This corresponds to having all the lines intersect ($Z_1 Z_2$). The positivity conditions then simply become

$$(x_i - x_j)(z_i - z_j) < 0 \tag{12.4}$$

which is trivial to triangulate. Whatever configuration of x 's we have are ordered in some way, say $x_1 < \dots < x_L$. Then we must have $z_1 > \dots > z_L$. The y_i just have to be positive. The associated form is then trivially (we omit the measure $\prod_i dx_i dz_i dy_i$):

$$\frac{1}{y_1} \dots \frac{1}{y_L} \frac{1}{x_1} \frac{1}{x_2 - x_1} \dots \frac{1}{x_L - x_{L-1}} \frac{1}{z_L} \frac{1}{z_{L-1} - z_L} \dots \frac{1}{z_1 - z_2} + \text{perm.} \tag{12.5}$$

Now, this cut is particularly simple to understand from the point of view of the familiar “local” expansions of the integrand — there is only one local diagram that can possibly contribute to this cut: the “ladder” diagram. The corresponding cut is precisely what we have above from positivity.



We can continue along these lines to explore faces of the amplituhedron which determine cuts to all loop orders that are difficult (if not impossible) to derive in any other way. For instance, suppose that some of the lines intersect ($Z_1 Z_2$), so that the $w_i \rightarrow 0$ for $i = 1, \dots, L_1$ and others intersect ($Z_3 Z_4$), so that $y_I \rightarrow 0$ for $I = L_1 + 1, \dots, L$. To pick a concrete interesting example, let choose $L - 2$ lines to intersect (12) and 2 lines to intersect (34). We can further specialize the geometry and take more cuts by making the L 'th line

pass through the point 3 — this corresponds to sending $z_L \rightarrow 0$. Let us also take the $(L-1)$ 'st line to pass through the point 4 — this corresponds to sending $z_{L-1}, w_{L-1} \rightarrow \infty$ with $w_{L-1}/z_{L-1} \equiv W_{L-1}$ fixed.

We can again label the $x_i; x_I$ so they are in increasing order; then the positivity conditions become

$$x_1 < \cdots < x_{L-2}, z_1 > \cdots > z_{L-2}; x_{L-1} < x_L \quad (12.6)$$

and

$$W_{L-1}y_i > (x_{L-1} - x_i), w_L y_i > z_i(x_i - x_L) \quad (12.7)$$

This space is also trivial to triangulate, but the corresponding form is more interesting. The ordering for the z 's is associated with the form

$$\frac{1}{z_{L-2}(z_{L-3} - z_{L-2})(z_{L-4} - z_{L-3}) \cdots (z_1 - z_2)}$$

The interesting part of the space involves x_i, y_i . Note that if $x_i < x_{L-1}$, the second inequality on y_i is trivially satisfied for positive y_i , and the only constraint on y_i is just $y_i > (x_{L-1} - x_i)/W_{L-1}$. If $x_{L-1} < x_i < x_L$, then both inequalities are satisfied and we just have $y_i > 0$. Finally if $x_i > x_L$, the first inequality is trivially satisfied and we just have $y_i > z_i(x_i - x_L)/w_L$. Thus, given any ordering for all the x 's, there is an associated set of inequalities on the y 's, and the corresponding form in x, y space is trivially obtained. For instance, consider the case $L = 5$, and an ordering for the x 's where $x_1 < x_4 < x_2 < x_5 < x_3$. The corresponding form in (x, y) space is just

$$\frac{1}{x_1(x_4 - x_1)(x_2 - x_4)(x_5 - x_2)(x_3 - x_5)} \frac{1}{y_1 - (x_4 - x_1)/W_4} \frac{1}{y_2} \frac{1}{y_3 - z_3(x_3 - x_5)/w_5} \quad (12.8)$$

By summing over all the possible orderings x 's, we get the final form. For general L , we can simply express the result (again omitting the measure) as a sum over permutations σ :

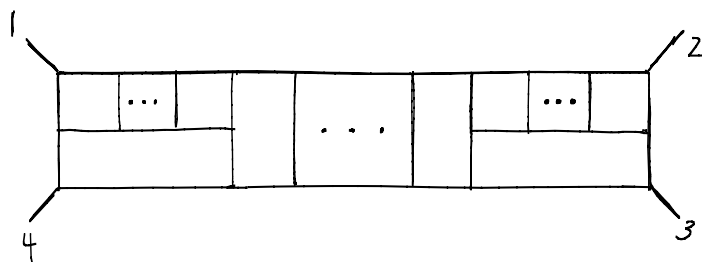
$$\prod_{l=1}^{L-2} \frac{1}{(z_l - z_{l+1})} \times \sum_{\sigma; \sigma_1 < \cdots < \sigma_{L-2}; \sigma_{L-1} < \sigma_L} \frac{1}{w_L W_{L-1}} \prod_{l=1}^L \frac{1}{(x_{\sigma_l^{-1}} - x_{\sigma_{l-1}^{-1}})} \quad (12.9)$$

$$\times \prod_{i=1}^{L-2} \left\{ \begin{array}{ll} (y_i - (x_{L-1} - x_i)/W_{L-1})^{-1} & \sigma_i < \sigma_{L-1} \\ y_i^{-1} & \sigma_{L-1} < \sigma_i < \sigma_L \\ (y_i - (x_i - x_L)z_i/w_L)^{-1} & \sigma_L < \sigma_i \end{array} \right\}$$

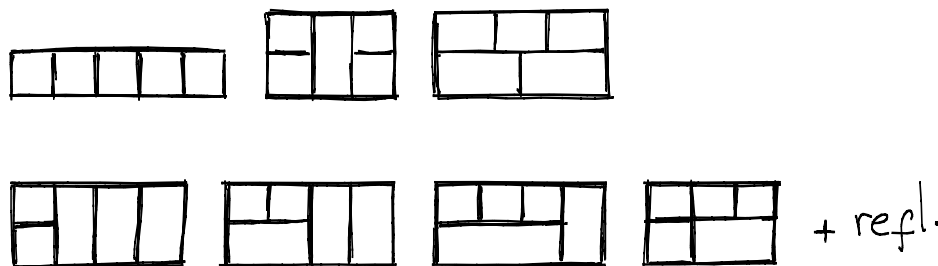
where we define for convenience $z_{L-1} = x_{\sigma_0^{-1}} = 0$.

This gives us non-trivial all-loop order information about the four-particle integrand. The expression has a feature familiar from BCFW recursion relation expressions for tree and loop level amplitudes. Each term has certain “spurious” poles, which cancel in the sum. This result can be checked against the cuts of the corresponding amplitudes that are

available in “local form”. The diagrams that contribute are of the type



but now there are non-trivial numerator factors that don’t trivially follow from the structure of propagators. The full integrand is available through to seven loops in the literature [36–40]. The inspection of the available local expansions on this cut does not indicate an obvious all-loop generalization, nor does it betray any hint that that the final result can be expressed in the one-line form given above. For instance just at 5 loops, the local form of the cut is given as a sum over diagrams,



with intricate numerator factors. If all terms are combined with a common denominator of all physical propagators, the numerator has 347 terms. Needless to say, the complicated expression obtained in this way perfectly matches the amplituhedron computation of the cut.

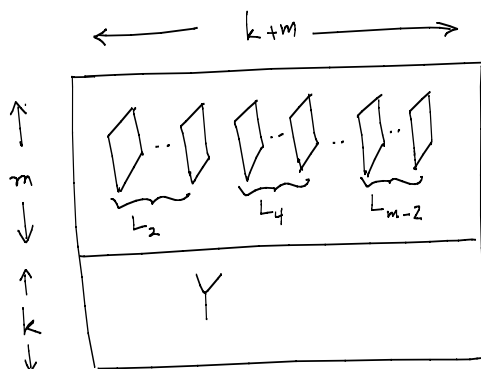
13 Master Amplituhedron

We have defined the amplituhedron $\mathcal{A}_{n,k,L}$ separately for every n, k and loop order L . However, a trivial feature of the geometry is that $\mathcal{A}_{n,k,L}$ is contained in the “faces” of $\mathcal{A}_{n',k',L'}$, for $n' > n, k' > k, L' > L$. The objects needed to compute scattering amplitudes for any number of particles to all loop orders are thus contained in a “master amplituhedron” with $n, k, L \rightarrow \infty$.

In this vein it may also be worth considering natural mathematical generalizations of the amplituhedron. We have already seen that the generalized tree amplituhedron $\mathcal{A}_{n,k,m}$ lives in $G(k, k+m)$ and can be defined for any even m . It is obvious that the amplituhedron with $m = 4$, of relevance to physics, is contained amongst the faces of the object defined for higher m .

If we consider general even m , we can also generalize the notion of “hiding particles” in an obvious way: adjacent particles can be hidden in even numbers. This leads us

to a bigger space in which to embed the generalized loop amplituhedron. Instead of just considering $G(k, k+4; L)$ of $(k-)$ planes Y together with L (2-planes) in $m = 4$ dimensional complement of Y , we can consider a space $G(k, k+m; L_2, L_4, \dots, L_{m-2})$, of k -planes Y in $(k+m)$ dimensions, together with L_2 (2-planes), L_4 (4-planes), $\dots L_{m-2}$ ($(m-2)$ -planes) in the m dimensional complement of Y , schematically:



Once again we have $\mathcal{Y} = \mathcal{C} \cdot \mathcal{Z}$, with the obvious extension of the loop positivity conditions on \mathcal{C} to $G(k, n; L_2, L_4, \dots, L_{m-2})$. We can call this space $\mathcal{A}_{n,k;m,L_2,\dots,L_{m-2}}(\mathcal{Z})$. The $m = 4$ amplituhedron is again just a particular face of this object. It would be interesting to see whether this larger space has any interesting role to play in understanding the $m = 4$ geometry relevant to physics.

14 Outlook

This paper has concerned itself with perturbative scattering amplitudes in gauge theories. However the deeper motivations for studying this physics, articulated in [14, 15] have to do with some fundamental challenges of quantum gravity. We have long known that quantum mechanics and gravity together make it impossible to have local observables. Quantum mechanics forces us to divide the world in two pieces — an infinite measuring apparatus and a finite system being observed. However for any observations made in a finite region of space-time, gravity makes it impossible to make the apparatus arbitrarily large, since it also becomes heavier, and collapses the observation region into a black hole. In some cases like asymptotically AdS or flat spaces, we still have precise quantum mechanical observables, that can be measured by infinitely large apparatuses pushed to the boundaries of space-time: boundary correlators for AdS space and the S-matrix for flat space. The fact that no precise observables can be associated with the inside of the space-time strongly suggests that there should be a way of computing these boundary observables without any reference to the interior space-time at all. For asymptotically AdS spaces, gauge-gravity duality [41] gives us a wonderful description of the boundary correlators of this kind, and gives a first working example of emergent space and gravity. However, this duality is still an equivalence between ordinary physical systems described in standard physical language, with time running from infinite past to infinite future. This makes the duality inapplicable to our universe for cosmological questions. Heading back to the early

universe, an understanding of emergent time is likely necessary to make sense of the big-bang singularity. More disturbingly, even at late times, due to the accelerated expansion of our universe, we only have access to a finite number of degrees of freedom, and thus the division of the world into “infinite” and “finite” systems, required by quantum mechanics to talk about precise observables, seems to be impossible [42]. This perhaps indicates the need for an extension of quantum mechanics to deal with subtle cosmological questions.

Understanding emergent space-time or possible cosmological extensions of quantum mechanics will obviously be a tall order. The most obvious avenue for progress is directly attacking the quantum-gravitational questions where the relevant issues must be confronted. But there is another strategy that takes some inspiration from the similarly radical step taken in the transition from classical to quantum mechanics, where classical determinism was lost. There is a powerful clue to the coming of quantum mechanics hidden in the structure of classical mechanics itself. While Newton’s laws are manifestly deterministic, there is a completely different formulation of classical mechanics — in terms of the principle of least action — which is not manifestly deterministic. The existence of these very different starting points leading to the same physics was somewhat mysterious to classical physicists, but today we know why the least action formulation exists: the world is quantum-mechanical and not deterministic, and for this reason, the classical limit of quantum mechanics can’t immediately land on Newton’s laws, but must match to some formulation of classical physics where determinism is not a central but derived notion. The least action principle formulation is thus much closer to quantum mechanics than Newton’s laws, and gives a better jumping off point for making the transition to quantum mechanics as a natural deformation, via the path integral.

We may be in a similar situation today. If there is a more fundamental description of physics where space-time and perhaps even the usual formulation of quantum mechanics don’t appear, then even in the limit where non-perturbative gravitational effects can be neglected and the physics reduces to perfectly local and unitary quantum field theory, this description is unlikely to directly reproduce the usual formulation of field theory, but must rather match on to some new formulation of the physics where locality and unitarity are derived notions. Finding such reformulations of standard physics might then better prepare us for the transition to the deeper underlying theory.

In this paper, we have taken a baby first step in this direction, along the lines of the program put forward in [14, 15] and pursued in [17–19]. We have given a formulation for planar $\mathcal{N} = 4$ SYM scattering amplitudes with no reference to space-time or Hilbert space, no Hamiltonians, Lagrangians or gauge redundancies, no path integrals or Feynman diagrams, no mention of “cuts”, “factorization channels”, or recursion relations. We have instead presented a new geometric question, to which the scattering amplitudes are the answer. It is remarkable that such a simple picture, merely moving from “triangles” to “polygons”, suitably generalized to the Grassmannian, and with an extended notion of positivity reflecting “hiding” particles, leads us to the amplituhedron $\mathcal{A}_{n,kL}$, whose “volume” gives us the scattering amplitudes for a non-trivial interacting quantum field theory in four dimensions. It is also fascinating that while in the usual formulation of field theory, locality and unitarity are in tension with each other, necessitating the introduction of the familiar redundancies to accommodate both, in the new picture they emerge together from positive geometry.

A great deal remains to be done both to establish and more fully understand our conjecture. The usual positive Grassmannian has a very rich cell structure. The task of understanding all possible ways to make ordered $k \times k$ minors of a $k \times n$ matrix positive seems daunting at first, but the key is to realize that the “big” Grassmannian can be obtained by gluing together (“amalgamating” [43]) “little” $G(1, 3)$ ’s and $G(2, 3)$ ’s, building up larger positive matrices from smaller ones [19]. Remarkably, this extremely natural mathematical operation translates directly to the physical picture of building on-shell diagrams from gluing together elementary three-particle amplitudes. This story of [19] is most naturally formulated in the original twistor space or momentum space, while the amplituhedron picture is formulated in momentum-twistor space. At tree-level, there is a direct connection between the cells of $G(k, n)$ that cellulate the amplituhedron, and those of $G(k+2, n)$, which give the corresponding on-shell diagram interpretation of the cell [19]. In this way, the natural operation of decomposing the amplituhedron into pieces is ultimately turned into a vivid on-shell scattering picture in the original space-time. Moving to loops, we don’t have an analogous understanding of all possible cells of the extended positive space $G_+(k, n; L)$ — we don’t yet know how to systematically find positive co-ordinates, how to think about boundaries and so on, though certainly the on-shell representation of the loop integrand as “non-reduced” diagrams in $G(k+2, n)$ [19] gives hope that the necessary understanding can be reached. Having control of the cells and positive co-ordinates for $G_+(k, n; L)$ will very likely be necessary to properly understand the cellulation $\mathcal{A}_{n,k,L}$. It would also clearly be very illuminating to find an analog of the amplituhedron, built around positive external data in the original twistor variables. This might also shed light on the connection between these ideas and Witten’s twistor-string theory [4, 44], along the lines of [45–48].

While cell decompositions of the amplituhedron are geometrically interesting in their own right, from the point of view of physics, we need them only as a stepping-stone to determining the form $\Omega_{n,k,L}$. This form was motivated by the idea of the area of a (dual) polygon. For polygons, we have another definition of “area”, as an integral, and this gives us a completely invariant definition for Ω free of the need for any triangulation. We do not yet have an analog of the notion of “dual amplituhedron”, and also no integral representation for $\Omega_{n,k,L}$. However in [27], we will give strong circumstantial evidence that such such an expression should exist. On a related note, while we have a simple geometric picture for the loop integrand at any fixed loop order, we still don’t have a non-perturbative question to which the full amplitude (rather than just the fixed-order loop integrand) is the answer.

Note that the form $\Omega_{n,k,L}$ is given directly by construction as a sum of “dlog” pieces. This is a highly non-trivial property of the integrand, made manifest (albeit less directly) in the on-shell diagram representation of the amplitude [19] (see also [49, 50]). Optimistically, the great simplicity of this form should allow a new picture for carrying out the integrations and arriving at the final amplitudes. The crucial role that positive external data played in our story suggests that this positive structure must be reflected in the final amplitude in an important way. The striking appearance of “cluster variables” for external data in [51] is an example of this.

We also hope that with a complete geometric picture for the integrand of the amplitude in hand, we are now positioned to make direct contact with the explosion of progress in

using ideas from integrability to determine the amplitude directly [52–55]. A particularly promising place to start forging this connection is with the four-particle amplitude at all loop orders. As we noted, the positive geometry problem in this case is especially simple, while the coefficient of the \log^2 infrared divergence of the (log of the) amplitude gives the cusp anomalous dimension, famously determined using integrability techniques in [56–58]. Another natural question is how the introduction of the spectral parameter in on-shell diagrams given in [59, 60] can be realized at the level of the amplituhedron.

On-shell diagrams in $\mathcal{N} = 4$ SYM and the positive Grassmannian have a close analog with on-shell diagrams in ABJM theory and the positive null Grassmannian [61], so it is natural to expect an analog of the amplituhedron for ABJM as well. Should we expect any of the ideas in this paper to extend to other field theories, with less or no supersymmetry, and beyond the planar limit? As explained in [19], the connection between on-shell diagrams and the Grassmannian is valid for any theory in four dimensions, reflecting only the building-up of more complicated on-shell processes from gluing together the basic three-particle amplitudes. The connection with the positive Grassmannian in particular is universal for any planar theory: only the measure on the Grassmannian determining the on-shell form differs from theory to theory. Furthermore, on-shell BCFW representations of scattering amplitudes are also widely available — at loop level for planar gauge theories, and at the very least for gravitational tree amplitudes (where there has been much recent progress from other points of view [62–67]). As already mentioned, one of the crucial clues leading to the amplituhedron was the myriad of different BCFW representation of tree amplitudes, with equivalences guaranteed by remarkable rational function identities relating BCFW terms. We have finally come to understand these representations and identities as simple reflections of amplituhedron geometry. As we move beyond planar $\mathcal{N} = 4$ SYM, we encounter even *more* identities with this character, such as the BCJ relations [68, 69]. Indeed even sticking to planar $\mathcal{N} = 4$ SYM, such identities, of a fundamentally non-planar origin, give rise to remarkable relations between amplitudes with different cyclic orderings of the external data. It is hard to believe that these on-shell objects and the identities they satisfy only have a geometric “triangulation” interpretation in the planar case, while the even richer structure beyond the planar limit have no geometric interpretation at all. This provides a strong impetus to search for a geometry underlying more general theories.

Planar $\mathcal{N} = 4$ SYM amplitudes are Yangian invariant, a fact that is invisible in the conventional field-theoretic description in terms of amplitudes in one space or Wilson loops in the dual space. We have become accustomed to such striking facts in string theory, which has a rich spectrum of U dualities, that are impossible to make manifest simultaneously in conventional string perturbation theory. Indeed the Yangian symmetry of planar $\mathcal{N} = 4$ SYM is just fermionic T -duality [70]. The amplituhedron has now given us a new description of planar $\mathcal{N} = 4$ SYM amplitudes which does not have a usual space-time/quantum mechanical description, but *does* make all the symmetries manifest. This is not a “duality” in the usual sense, since we are not identifying an equivalence between existing theories with familiar physical interpretations. We are seeing something rather different: new mathematical structures for representing the physics without reference to standard physical ideas, but with all symmetries manifest. Might there be an analogous story for superstring scattering amplitudes?

Acknowledgments

We thank Zvi Bern, Jake Bourjaily, Freddy Cachazo, Simon Caron-Huot, Clifford Cheung, Pierre Deligne, Lance Dixon, James Drummond, Sasha Goncharov, Song He, Johannes Henn, Andrew Hodges, Yu-tin Huang, Jared Kaplan, Gregory Korchemsky, David Kosower, Bob MacPherson, Juan Maldacena, Lionel Mason, David McGady, Jan Plefka, Alex Postnikov, Amit Sever, Dave Skinner, Mark Spradlin, Matthias Staudacher, Hugh Thomas, Pedro Vieira, Anastasia Volovich, Lauren Williams and Edward Witten for valuable discussions. Our research in this area over the past many years owes an enormous debt of gratitude to Edward Witten, Andrew Hodges, and especially Freddy Cachazo and Jake Bourjaily, without whom this work would not have been possible. N. A.-H. is supported by the Department of Energy under grant number DE-FG02-91ER40654. J.T. is supported in part by the David and Ellen Lee Postdoctoral Scholarship and by DOE grant DE-FG03-92-ER40701 and also by NSF grant PHY-0756966.

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