

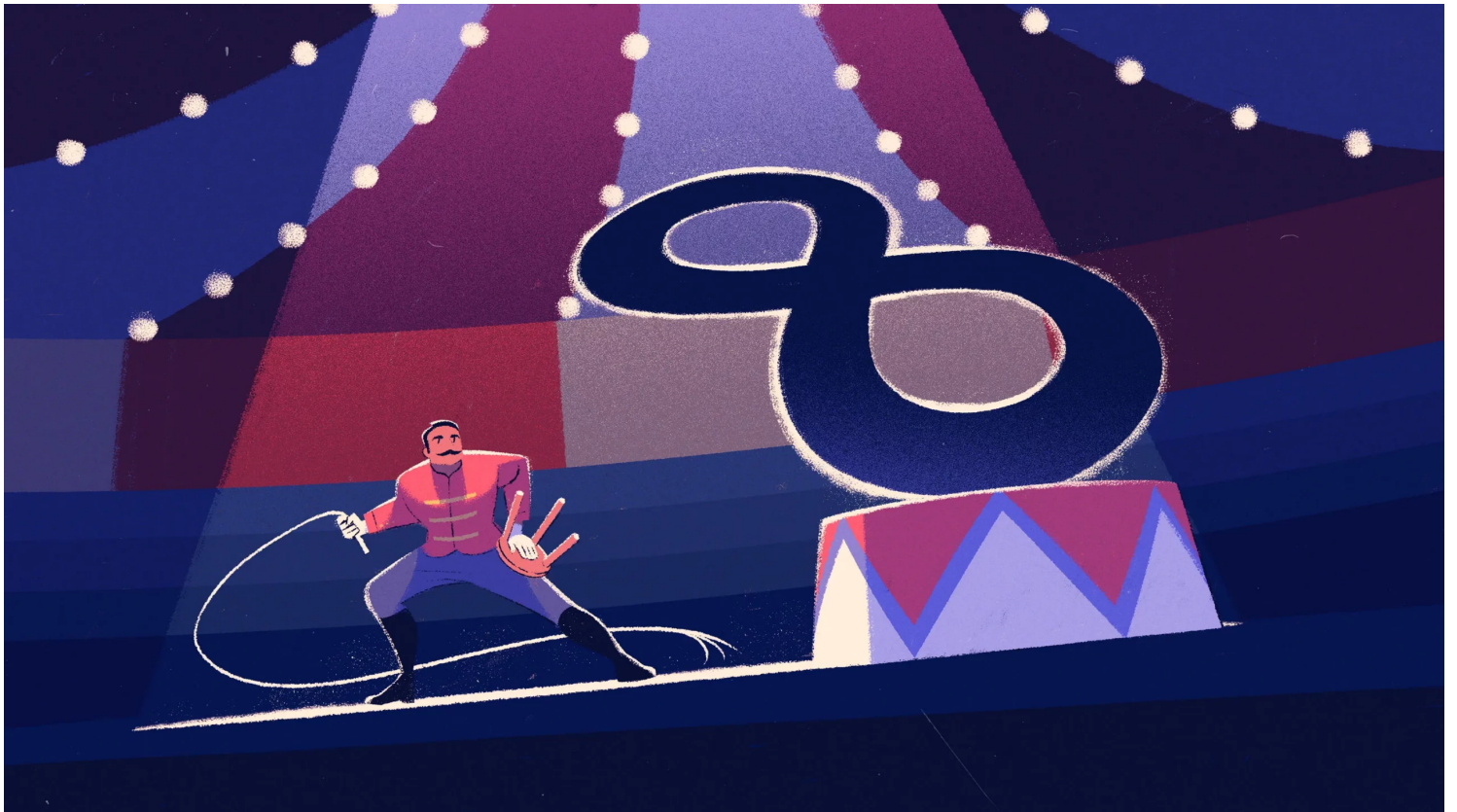
## MATHEMATICAL PHYSICS

**How to Tame the Endless Infinities Hiding in the Heart of Particle Physics**

By CHARLIE WOOD

April 6, 2023

*In the math of particle physics, every calculation should result in infinity. Physicists get around this by just ignoring certain parts of the equations — an approach that provides approximate answers. But by using the techniques known as “resurgence,” researchers hope to end the infinities and end up with perfectly precise predictions.*



---

Tommy Parker for Quanta Magazine

**T**he secret to fixing a fatal flaw at the heart of quantum theory may lie in three obscure textbooks from the 1980s. But physicists can be forgiven for overlooking the potentially transformative ideas within, as the volumes appear simultaneously amateurish and intimidating.

The few physical copies that exist of Jean Écalle's magnum opus look like little more than glorified photocopies. Oversize mathematical symbols scrawled in thick black ink frequently interrupt the neatly typed sentences. The text is also written in French, an inconvenience for researchers in the English-speaking world.

The mathematics itself poses another barrier. The trilogy's 1,110 pages brim with original mathematical objects and bizarre coinages. Odd-sounding terms like "trans-series," "analyzable germs," "alien derivations" and "accelero-summation" abound.

"If you have a look at this for the first time and you don't read it very carefully, you could think it's a crackpot writing some crazy things," said [Marcos Mariño](#), a mathematical physicist at the University of Geneva who keeps what he calls the "historical documents" on his bookshelf and uses tools developed by Écalle daily. "Of course, he is not. He's one of these visionary mathematicians."

His visionary mathematics might be just what's needed to overcome a profound conceptual embarrassment — one that physicists have been more or less ignoring for the past 70 years. In that time, physicists have learned to make breathtakingly accurate predictions about the subatomic world. But these predictions, precise though they may be, are approximations. If one seeks absolute precision, textbook quantum theory breaks down and yields infinite answers — nonsensical results many physicists consider to be mathematical trash.

By studying Écalle's vintage textbooks, physicists are coming to suspect that these infinite answers contain countless treasures, and that, with sufficient effort, the mathematical tools he developed should let them to take any infinity and dig out a finite and faultless answer to any quantum question.

"Indeed, it works very beautifully" in many cases, said [Marco Serone](#), a physicist who studies this strategy, which goes by the name of "resurgence." "At some point this process ends, and what you have in front of your eyes is the exact solution to your original problem."

The resurgence community is small but has made steady progress over the years. A proto-version of the technique obtained exact results in quantum mechanics, which limits itself to the behavior of particles. And more sophisticated incarnations have allowed some physicists to venture further into the murky waters of quantum field theory, and recently string theory. But that's just the beginning of the big dreams harbored by resurgence practitioners. They aim at nothing less than a new way of thinking about infinities in physical theories — one that better matches our finite world in theory and, just maybe, in practice too.

### **Exploding Possibilities**

Quantum field theory — the notion that particles like electrons are really sustained ripples in an underlying quantum field — forced postwar physicists to face infinity head on.

These quantum fields are unimaginably complicated beasts — with transient ripples and coherent waves roiling seemingly empty space. These passing ripples can, in principle, appear at any moment, in any number, and with any energy — challenging physicists to account for an unending array of subatomic mingling in order to understand the precise outcome of even simple experiments.

In the 1940s, Shin'ichirō Tomonaga, Julian Schwinger and Richard Feynman all worked out equivalent ways of getting finite answers out of the infinite complexity of the quantum electromagnetic field. Best known today in Feynman's presentation, the calculation took the form of an infinite string of "Feynman diagrams" representing a parade of increasingly byzantine quantum possibilities. You start with the diagram for the simplest possible event — an electron moving through space, say — and calculate some measurable property, such as how much the electron wobbles in a magnetic field. Next you add the result from a more complicated scenario, such as the electron briefly expelling and then reabsorbing a photon on the fly. You then add subatomic drama involving two transient ripples, then three and so on, in a widely used mathematical technique known as perturbation theory.

On paper, a calculation of this property creates a never-ending “power series”: an equation that involves a certain critical value, which we’ll call  $x$ , then  $x$  squared,  $x$  cubed, and higher and higher powers of  $x$ , all multiplied by different coefficients:

$$F(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_{1,000,000}x^{1,000,000} + \dots$$

For the electromagnetic field, the value of  $x$  is a linchpin constant of nature,  $\alpha$ , which is close to  $1/137$ . It’s a small number befitting the force’s relative weakness, and raising this tiny number to larger powers makes the terms shrink rapidly.

Feynman diagrams give physicists the coefficients for each term — the  $a$ ’s — which are the hard parts to calculate. Take the calculation of the electron’s “ $g$ -factor,” a number related to the way the particle wobbles in a magnetic field. The simplest Feynman diagram gives you  $a_0$ , which is equal to exactly 2. But if you consider a slightly more complicated Feynman diagram, one where the first temporary ripple pops up, you need to calculate the  $a_1$  term, and that’s where infinity rears its head. Tomonaga, Schwinger and Feynman worked out a way to make this term finite. Their calculation of about 2.002 for the electron’s  $g$ -factor matched that generation’s experimental measurements, proved that quantum field theory could make sense, and earned the three of them the 1965 Nobel Prize in Physics.

Their approach also launched a new era, one where physicists had to scale ever-taller mountains of Feynman diagrams in order to calculate more  $a$ ’s. Those mountains get steep, and fast. In 2017, a physicist finished a two-decade labor of love, a precise calculation of the electron’s  $g$ -factor that required computing hairy equations from 891 Feynman diagrams. The result revealed just the fifth term in the series.

Feynman diagrams remain critically important in modern physics. A collection of similar but even more involved calculations for the muon, the electron’s portly cousin, made headlines in 2021. An experiment revealed an eighth-decimal discrepancy from theoretical predictions. The modest anomaly represents one of the best hopes for seeing what lies beyond the towering edifice that has grown from the work of Feynman and his colleagues.

But this string of experimental victories has hidden the fact that, deep down, this way of approaching quantum field theory doesn’t really work at all.

### **The Fall of Feynman Diagrams**

Freeman Dyson, another postwar pioneer, was the first physicist to appreciate that perturbative quantum theory was probably doomed. The year was 1952, and while others were celebrating the fact that the first couple of terms in Feynman’s power series could be made small and finite, Dyson was worrying about the rest of the series.

Physicists were naïvely hoping that the Feynman diagram treatment of the electromagnetic field would turn out to be what mathematicians call “convergent.” In a convergent series, each subsequent term is much smaller than the previous term, and the more terms there are, the more the sum converges to a single finite number. In contrast, a series can also be “divergent” — later terms are bigger than earlier terms and the series grows without limit. The sum “diverges,” giving no obvious meaningful answer.

The first terms of Feynman's sum did indeed shrink — a consequence of the tiny value of alpha — and Dyson himself at first concluded that perturbative quantum electromagnetism should be convergent overall.

But then Dyson blended mathematical and physical reasoning to make a more sophisticated guess about the fate of the series. Thinking mathematically, Dyson knew that a convergent power series converges faster when  $x$  becomes smaller, because the higher terms (which involve powers of  $x$ ) shrink more quickly.

But when he allowed  $x$  to pass through zero, everything fell apart.

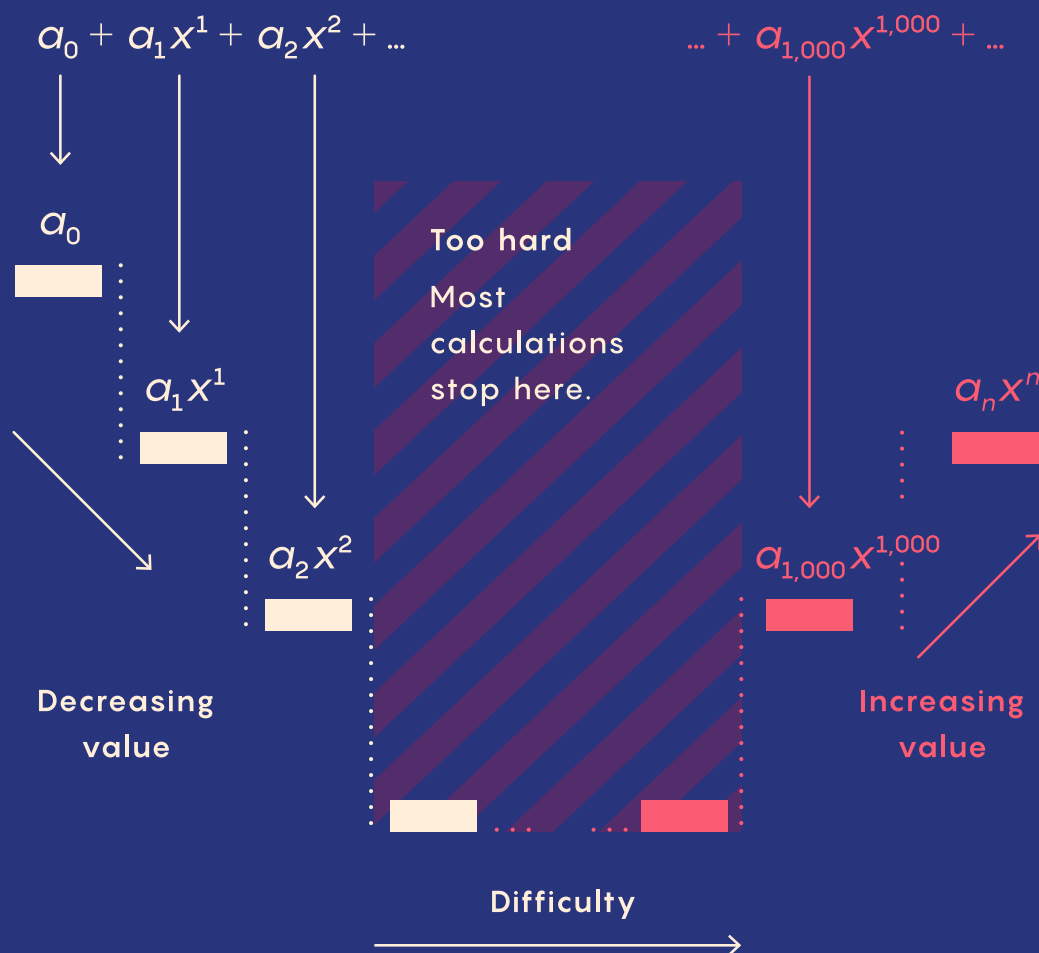
The reason has to do with our vacuum, which constantly produces transient pairs of ripples with positive and negative charges. Those ripples normally attract each other and vanish. But if alpha were to become negative, those ripples would push each other apart and become real particles. The continuous eruption of particles from nothing would trigger a cosmic meltdown, an “explosive disintegration of the vacuum,” as Dyson put it.

Physically, any negative alpha is trouble. Yet mathematically, the sign of  $x$  is irrelevant: If a series diverges for a small negative  $x$  then it should also diverge for a small positive  $x$ . Therefore, for a small positive alpha (namely,  $1/137$ ), the series should also diverge. Dyson's catastrophic physical situation implied that Feynman's celebrated way of handling of quantum electromagnetism predicted, eventually, infinity.

## An Untamed Infinity

Physicists usually study quantum fields, such as the electromagnetic field, using unending sums. The first few terms in these sums get smaller and smaller, providing an approximate answer. Later terms explode, and the sum seems to lose all meaning.

### Example sum



Merrill Sherman/Quanta Magazine

Today, physicists expect quantum electrodynamics (as the quantum field theory of electromagnetism is called) to start diverging somewhere around the 137th term. That is, perhaps,  $a_{138}x^{138}$  may be larger than  $a_{137}x^{137}$ , and including it in the sum will make the prediction less — rather than more — precise.

The problem is that higher terms lead to explosive growth — factorial growth — in the number of Feynman diagrams. That means calculating  $a_9$  will require roughly  $9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$  (about 362,880) diagrams, and  $a_{10}$  will call for around  $10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$  (3,628,800) diagrams. This factorial growth in the diagrams contributing to the  $a$ 's will eventually beat the shrinking of the powers of alpha, and the sum will grow untamed toward infinity.

To most physicists, the inevitable divergence of even the simplest quantum field theory remains an abstract problem, like the death of our sun in a billion or so years. At a time when calculating — much less testing — even the 10th term of the series seems like science fiction, why fret about dangers lurking far beyond the 100th?

But to a select few, the fact that the best-understood theory in modern physics technically yields infinite answers to any question you might care to ask remains deeply disturbing. “We do not know how to simulate the world, even in principle, even with unlimited computational resources,” said Emanuel Katz, a physicist at Boston University who studies new methods for going beyond Feynman diagrams.

### The Devil's Divergence

Mathematicians, meanwhile, had been puzzling over divergent series for more than a century before Dyson started fretting about quantum theory.

“Divergent series are the invention of the devil, and it is shameful to base any demonstration on them whatsoever,” quipped Niels Henrik Abel in 1828. “For the most part the results are valid, it is true, but it is a curious thing. I am looking for the reason.”

Abel died the next year, aged 26. But near the end of the century Henri Poincaré took a significant step toward understanding what made divergent series so slippery: They weren't satanic, just incomplete.

Poincaré was picking at an age-old question: How might three celestial bodies orbit each other? He set out to tackle the problem using perturbation theory, just as Feynman and Dyson would do when they encountered quantum fields a century later. Poincaré sought to construct the mysterious, presumably complicated function that describes the trajectories of the three bodies using an infinitely long sum of simpler units — a process akin to building a car out of simple Lego pieces. The hope was that the series would converge to a finite answer, a sign that the series was a perfect representation of a unique function.

Initially, he thought he had succeeded. In 1890, King Oscar II of Sweden and Norway awarded Poincaré a prize for his progress on the famous problem. But shortly before his solution was to be published, he called for the king to stop the presses. The series was divergent. Further analysis (which would lay the foundation for chaos theory) revealed that it matched not one but two distinct functions. It was a complication that physicists are now all too familiar with.



Carl Bender, a mathematical physicist at Washington University in St. Louis, has calculated many terms of perturbative series in physics and studied their behavior.

—

Courtesy of Carl Bender



“It would be a complete miracle if your physics problem that you’re interested in is actually associated with a convergent series,” said Carl Bender, a prominent mathematical physicist at Washington University in St. Louis. (Today, physicists know that three celestial bodies can interact in a countless number of very different ways, and no simple equation can contain all the possibilities.)

Bender likens the sort of divergent series that Poincaré encountered to a blurry view of a function. The blur accommodates many possible functions, much as the blocky silhouette of a Lego vehicle could match any number of sports cars. When you expand a complicated function into such an “asymptotic” series, “you have lost information,” Bender said.

Since the days of Poincaré, mathematicians and physicists have come to appreciate that there are other types of terms, ones that are “beyond all orders,” that are even tinier than the tiniest power term. These “exponentially small” terms can come in the form of  $e^{(-1/x)}$ , for instance, and they supply the lost information. If you include them in your series and select an appropriate “resumming” procedure to make the series finite, you can get rid of some — if not all — of the blur. They are the nano-Lego blocks needed to tell a Ferrari from a Lamborghini.

Physicists call these extra terms “nonperturbative,” because they are beyond the reach of perturbation theory. You can spend a trillion years drawing Feynman diagrams and calculating  $a$ ’s, and you’ll never learn about certain physical events encoded in these nonperturbative terms. While the effects described by these tiny terms may be rare or subtle, they can make a dramatic difference in the real world.



Take the Schrödinger equation of quantum mechanics, for instance, which describes the wavelike behavior of particles. It's a complicated equation that physicists often approximate using perturbation theory. Although the resulting infinite series beautifully predicts many experiments, it completely misses an extremely unlikely (but not impossible) event known as tunneling, in which the particle essentially teleports through a barrier.

Tunneling is one of many nonperturbative phenomena in quantum physics, but nonperturbative effects are everywhere: The branching growth of snowflakes, the flow of a liquid through a pipe with holes, the orbits of planets in a solar system, the rippling of waves trapped between round islands, and countless other physical phenomena are nonperturbative.

"They are there, and they are crucial," said Daniele Dorigoni, a physicist at Durham University. "Perturbation theory on its own is not enough."

Due to its universal nature, hordes of mathematicians and physicists have worked on various aspects of the meta-problem of how to calculate nonperturbative terms. And toward the end of the 20th century, an assortment of researchers started finding tantalizing hints that perturbative series seemed to know more than they should.

Among these researchers, a group at the Saclay Nuclear Research Center in France in the 1980s helped develop a way of combining perturbative power terms with nonperturbative exponential terms to get exact results for tunneling in quantum mechanics. Their technique worked insofar as they could rely on a crucial mathematical technology from the turn of the century known as Borel resummation. Borel resummation was the most powerful tool of the day for getting finite numbers out of divergent series, but it had its limits. It occasionally gave wrong or conflicting results, frustrating physicists who hoped one series would correctly predict the outcome of one experiment.

“When physicists found a series which was not Borel summable, they would essentially give up,” Mariño said.

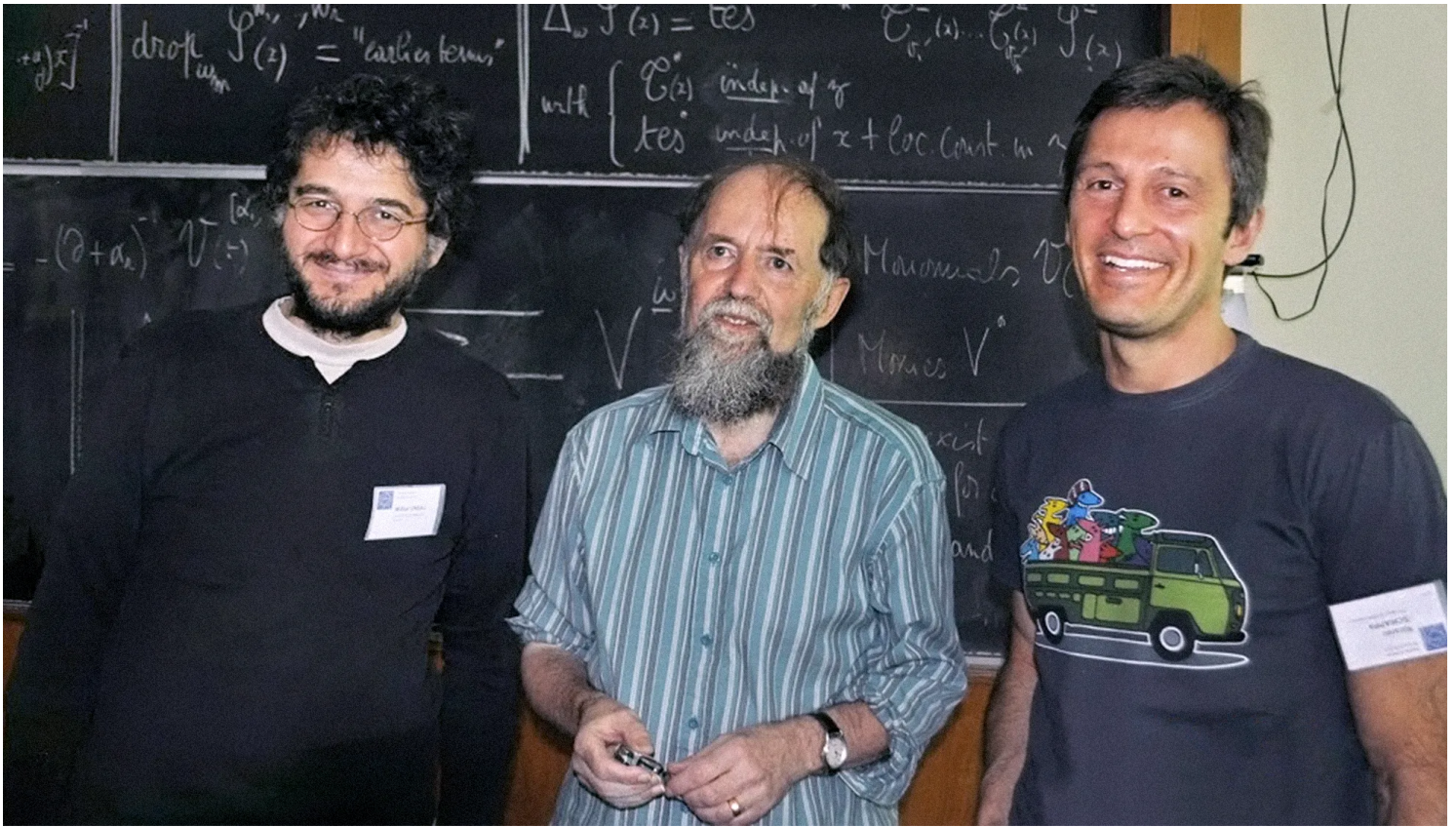
Unbeknownst to them, an eccentric mathematician working in isolation just miles from the group at Saclay had already mounted an unprecedented exploration of the infinitely tall peaks of asymptotic series.

### **Feynman Diagrams Strike Back**

Jean Écalle has felt captivated by the mathematics of infinity since he was a teenager. He recalls relaxing on the banks of a mountain stream one summer in high school and wondering whether there might be a more general version of the derivative operation — an exercise in infinitesimals that students first learn in elementary calculus.

As he continued his education, Écalle developed a taste for working alone. He even tried to avoid reading the work of his fellow mathematicians, for fear that their thinking would pull him into established ruts.

“I am temperamentally averse to losing myself in the math literature,” Écalle said. “I could also observe, time and again, how too deep an immersion in the math literature tended to stifle creativity.”



Jean Écalle, the father of modern resurgence theory, flanked by two of its leading practitioners — Mithat Ünsal (left) and Ricardo Schiappa — in 2014.

—  
Mikhail Shifman

In the early 1970s, Écalle's curiosity drove him to follow in the footsteps of Poincaré. He began to analyze even more abstract mathematical objects that arose in the study of celestial bodies. Asymptotic series cropped up along the way, as did the more general derivative he had speculated about back in high school. Écalle would eventually develop what he described as "a precise, sharp-contoured structure — alien calculus — spontaneously arising from what would seem to be the most unpromising and amorphous of contexts: divergence."

Écalle's alien calculus is abstract and multifaceted. But the message it held for the physicists that would eventually encounter it was clear. A perturbative series, even though it diverges, hides a complete library of nonperturbative information. The series contains all that's needed to upgrade it in a way that removes the blur, restoring a crisp image of a unique corresponding function. The blocky Lego bricks, perhaps, are enough after all.

Despite its deep consequences, Écalle's work languished at first. It was too obscure and too abstract for physicists (even the French-speaking ones). And it wasn't rigorous enough to catch the eye of mathematicians.

"He is one of these geniuses who thinks that the detailed proofs, with all the cases, are not important. What's really important is the grand view," Mariño said.

Écalle first sketched out the core concepts of resurgence in three papers in 1976, and between 1981 and 1985 he penned his three textbooks, in which he thoroughly laid out the alien calculus of resurgence. They never appeared in a mathematical journal. Instead, he published the trilogy through his university's mathematics department, filling in equations by hand.

If physicists had managed to dig into his books right away, their experience would not have been unlike contact with an intelligent extraterrestrial civilization. They would have encountered mathematical machinery light-years ahead of what they were used to.

"Resurgence is very fancy," Bender said. But, to put it as simply as possible, it lets practitioners dig into the distant terms of an asymptotic series (calculated using Feynman diagrams, for instance) and uncover the missing pieces necessary to specify a unique function (one that describes tunneling, say). In short, it reveals a bridge linking physical events described by perturbation theory with those described by the nonperturbative terms. "It's a very complicated relationship," Bender said, before politely declining to attempt to explain it.

When Écalle, now 73, was contacted by *Quanta Magazine* with questions about the history of resurgence, he responded by composing a [24-page treatise](#) on the subject in six days — a treat for researchers hungry for more information about resurgence and its development. "It's a treasure," said David Sauzin, a mathematician at the Institute of Celestial Mechanics in Paris and renowned Écalle decoder.

Here's an extremely rough cartoon version of the approach:

First, write out the typical perturbative series. The terms shrink at first, but eventually they grow rapidly as the  $a$ 's get really big. Plot the growth of the  $a$ 's, and you will see that they shoot upward with a speed that nearly — but not exactly — matches factorial growth. Study the difference between the

line traced out by the  $a$ 's and a curve growing factorially to learn the first nonperturbative term — the largest of the nano-Lego bricks.

But that's just the beginning. Apply the first step of a Borel resummation. This eliminates the factorial growth, allowing you to see the behavior of the perturbative terms in more detail. The resulting plot of modified  $a$ 's should grow exponentially. But study it carefully, and you'll see that the perturbative data is a little off. This deviation comes from a whole new asymptotic series, which you multiply by the first nonperturbative term.

The procedure continues. Strip the exponential growth from the perturbative data, and if you have a keen eye, you may spot further deviations that reveal a second nonperturbative term. Look closer, and you'll find that this nonperturbative term comes with yet another asymptotic series.



Marcos Mariño, a mathematical physicist at the University of Geneva, uses resurgence to explore string theory and quantum field theory.

---

Anna Serra Picamal

At the end of the day, there may be any number of nonperturbative terms with asymptotic series attached. Find as many of these as you have the stomach for, and you'll have an object called a trans-

series on your hands. The trans-series starts out with the familiar perturbative series. Then comes a nonperturbative term (with a series), and then another and another.

Écalle's trans-series overcame the difficulties with Borel resummation that had previously stumped physicists. If you know the trans-series describing some measurement, such as the electron's  $g$ -factor, Borel resummation will get you a single, correct answer. Moreover, resurgence asserts that subtle deviations in the familiar perturbative series at the head of the trans-series tell you everything you need to know about the potentially infinite parade that follows.

This mathematical picture has two striking consequences for physicists. First, it suggests that exact results — not merely approximations — could exist for quantum fields and other complicated systems. If so, it would establish quantum theory as finite and sensible.

"Establishing that in quantum field theory indeed things are subject to resurgence would be a major advance," Serone said.

Second, it suggests that the potentially infinite assortment of nonperturbative pieces can be deduced entirely from the perturbative series whose divergence troubled Dyson. What for decades seemed like independent realms of physics are in fact intimately related.

"Instead of thinking about the perturbative series as something that's going to diverge and give you a bunch of trouble," Mariño said, "it's just the entrance to a very complex and fascinating world."

Indeed, that's where the name resurgence comes from, said [Gökçe Başar](#), a physicist at the University of North Carolina, Chapel Hill: "The behavior of the late terms in the perturbative series 'resurges' in those nonperturbative terms." It's convoluted, he said, but "it's rather beautiful."

### Surging Into Physics

Awareness of Écalle's discovery — that nonperturbative knowledge could be secretly accessed through perturbation theory — has slowly trickled into the world of mathematical physics. There, physicists have already used it to identify new pieces hidden in two of the most intensely studied theories of the 21st century: the theory of the strong force, and string theory.

[Mithat Ünsal](#), a physicist at North Carolina State University, has devoted much of his career to trying to understand the strong force, which holds quarks together to form protons and other particles. In 2008, after reading about resurgence in a [1993 article](#) about divergent series, he sought out an overview of Écalle's work. "My French is very rusty, but there was an English preface with suggested terminology," Ünsal recalled. "I mastered it and tried to understand it."

He later met [Gerald Dunne](#) of the University of Connecticut at a conference, and while chatting over coffee they discovered that the same article had inspired them both to start teaching themselves resurgence. They decided to join forces.

Both physicists were motivated by the fact that they were trying to understand something even more complicated than what confronted Dyson and Feynman. Those physicists lucked out with the electromagnetic field. It's extremely weak, with  $\alpha$  being just  $1/137$ . Another fundamental force, the



weak interaction, proved similarly easy to tame, with its version of alpha being 10,000 times smaller still. Perturbation theory happens to work for these two forces because they are so feeble that it's almost as if they don't exist at all.



Gerald Dunne, a physicist at the University of Connecticut, is searching for ways to make resurgence practical for quantum field theory calculations.

—

Bri Diaz

But that luck ended when physicists tried to tackle the strong force. The strong force is around 100 times stronger than the electromagnetic force, with an alpha analogue of about 1, and it refuses to be ignored. Squaring or cubing 1 doesn't create any shrinking effect whatsoever, so the perturbative series heads straight toward infinity from the earliest terms. Physicists have spent decades developing an alternative way to handle the strong force using supercomputers, achieving spectacular results along the way. But the numerical calculations don't give much insight into how the strong force does what it does.

Ünsal and Dunne recognized that resurgence, with its power to tame divergent series, could take them a step toward the dream of understanding the strong force with pencil and paper. In particular, they set out to solve a mystery that had plagued the theory of the strong force for 40 years.

In 1979, the physicists Gerard 't Hooft and Giorgio Parisi inferred the existence of tiny, bizarre terms in strong force calculations. They called them renormalons, and no one knew what to make of them. Renormalons didn't seem to correspond to any specific ripple or other concrete field behavior. But there they were, messing up the calculations nonetheless.

Ünsal and Dunne tackled renormalons with resurgence. Even though they were working in a 2D analogue of the strong force, it took them roughly a year. But in 2012, they showed that — at least in their simplified model — 't Hooft and Parisi's renormalons matched behaviors that physicists understood.

They “solved the mystery and could find what it was that the renormalons corresponded to,” said Jordan Cotler, a physicist at Harvard University who is currently mounting a similar attempt to understand renormalons in a more realistic theory of the strong force.

Last year, however, researchers used resurgence to add a further wrinkle. Mariño and his collaborators carried out a more rigorous calculation (although also in a simplified theory) and discovered new renormalons beyond what the group calls “the standard lore” of 't Hooft and Parisi. Mariño now suspects that renormalons are just the tip of a nonperturbative iceberg. Resurgence and other nonperturbative methods may reveal that physicists have been spoiled by their historical success in matching individual mathematical terms to specific events. If he's right, the quantum world may someday become even harder to visualize than it already is.

“I have doubts that this picture — one exponential [to] one object — is going to go through in general field theories,” he said. “It might happen that the world of exponential corrections is really wild.”

Mariño has also been a key player in the discovery of a new nonperturbative effect in string theory, the speculative and unproved notion that the universe is not made of pointlike particles but is composed of extended objects such as strings. The wiggling of such strings would determine the properties of the particles we observe.

String theory, like quantum theory, is usually treated as a perturbative series of Feynman-like diagrams representing strings merging and splitting in increasingly complicated ways. But unlike quantum theorists, string theorists lack even the faintest of guides to the theory's nonperturbative effects. They assume that, just as quantum theory contains tunneling and renormalons, the full nonperturbative formulation of string theory also contains dragons.

One striking example of non-perturbative phenomena in string theory — sheetlike objects known as D-branes — was discovered in the 1990s. D-branes would later spur some of string theory's biggest developments.

Mariño wondered what else might be out there.

He was part of a group that in 2010 noticed a series of negative counterparts hiding in the shadow of the D-brane terms. It wasn't clear what physical phenomenon these partner terms might describe.

A clue came six years later, when [Cumrun Vafa](#) of Harvard and his collaborators explored a generalized string theory where certain quantities could go negative. They found D-branes with negative tension — the brane version of having negative mass. [These exotic beasts](#) warped the structure of reality around them, creating multiple dimensions of time and violating the fundamental principle that probabilities must always add up to 100%. But the group found no indication that these objects should escape from their bizarre world and show up in standard string theory.

Now [Ricardo Schiappa](#), a friend of Mariño's and a theoretical physicist at the University of Lisbon, believes he's found evidence otherwise. In recent months, Schiappa and his collaborators used resurgence to scrutinize a handful of simple string theory models. They found that Vafa's negative-tension D-branes exactly matched the exponentially small terms that Mariño had found in 2010. Negative D-branes are unavoidable partners of D-branes, the group argued in a [January preprint](#). "What we have discovered now is that they are fundamental for perturbation theory," Schiappa said.

Other theorists aren't yet sure what to make of the fresh finding. Vafa notes that Schiappa's crew did their calculations in stripped-down string models, and that the result isn't guaranteed to hold in more sophisticated formulations. But if it does, and if string theory actually describes our universe, it must contain some other way of stopping negative D-branes from forming.

"They shouldn't be there as a regular object in that theory," Vafa said. Otherwise, "this opens a whole Pandora's box of puzzles."

### **Black Swans and Other Anomalies**

Despite their progress in spotting renormalons and negative branes, physicists cite two formidable obstacles to crowning resurgence the official successor to perturbation theory.

First, not all theories have been proved to have resurgent structure. The question is particularly acute for quantum field theories, which physicists have been checking on a case-by-case basis. It's a painstaking process, a bit like studying mammals one species at a time. After observing humans, dolphins and cats, you might start to feel confident that live birth is a universal mammalian feature. But there's always the chance that around the next corner you'll find a platypus laying an egg.

That's why Serone has devoted the last three years to stress-testing resurgence in certain quantum field theories. In 2021, he and his collaborators [studied a theory](#) that shares key features with the strong force but is still simple enough to allow them to calculate the many  $a$ 's needed to perform resurgence. They calculated the energy of empty space in such a universe using resurgence and two other methods, showing that all three agreed. There have been qualitative arguments that resurgence

should hold in quantum field theory, but this was one of the first concrete calculations, kindling further optimism.

“In most of the cases it has been tested so far, either resurgence works, or we have good reasons to believe we understand when it doesn’t,” Serone said.

The graver problem is that to spot nonperturbative pieces, you need to know a frightening number of perturbative terms. In his recent research, for instance, Serone picked quantum field theories with mathematical backdoors that let him generate thousands of terms. But for the strong force, calculating just eight or nine is currently out of the question. Even pioneers of the method don’t mince their words about when they expect to see it produce a real number like the mass of the proton (a mathematical feat worth a million-dollar prize).

“It’s extremely difficult,” Ünsal said, sighing. “I don’t see an immediate way.”

“What Écalle was saying is that the answer is rigorously there in principle. But to actually get the answer is really, really hard,” Bender said. “My advice would be, don’t stand on one foot while you’re waiting.”

### A New Hope

But the daunting difficulty hasn’t killed the dream of trying to get real predictions out of resurgence. For one thing, the technique has already produced otherwise unobtainable results in quantum mechanics. Back in the 1980s, the French mathematical physicists at Saclay used proto-resurgent methods to make an exact prediction for particle tunneling — a problem that physicists had previously only been able to approximate. Dunne and Ünsal have done similar pen-and-paper calculations using the more refined tools of Écalle. Another group has checked these results using standard methods. They were only able to get as far as six decimal places — a Herculean effort that took months of time and substantial computer power.

Such dramatic examples have motivated Dunne to develop hyper-efficient ways of practicing resurgence, in the hopes of someday porting them to quantum field theories. Over the last five years, together with Ovidiu Costin, a mathematician at Ohio State University, he has found techniques that get more bang for the perturbative buck. In some cases (which are still far from the real-world theories), they’ve found that just 10 to 15 terms suffice. “That number could have come out to be 1,000, and I would have given up and gone somewhere else,” he said. “It’s kind of tantalizing.”

Dunne and Costin’s work has even managed to catch the eye of Écalle himself. The founder of resurgence hasn’t closely followed the waves his work set off, calling himself “an accomplished ignoramus in theoretical physics.” Nevertheless, while worrying that any work on speculative models such as string theory may be “built on quicksand,” he praises the researchers’ efforts to give resurgence a mathematical tune-up.

“Even if the physical ground gives way, the impressive math results of, say, O. Costin and G. Dunne are there to stay,” he said.

For Écalle, resurgence is something of a past chapter. Nearly 40 years have passed since his original trilogy. He continued to develop alien calculus until around 2000, and he has spent the last 20 years exploring a more algebraic offshoot. Should he ever decide to publish a sequel trilogy gathering all his findings in one place, who knows what treasures physicists will find within.

“I think he has discovered many tools that are still to be explored,” Mariño said.