QUANTIZED COLUMNS

# How Math Achieved Transcendence

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*Transcendental numbers include famous examples like e and $\pi$, but it took mathematicians centuries to understand them.*



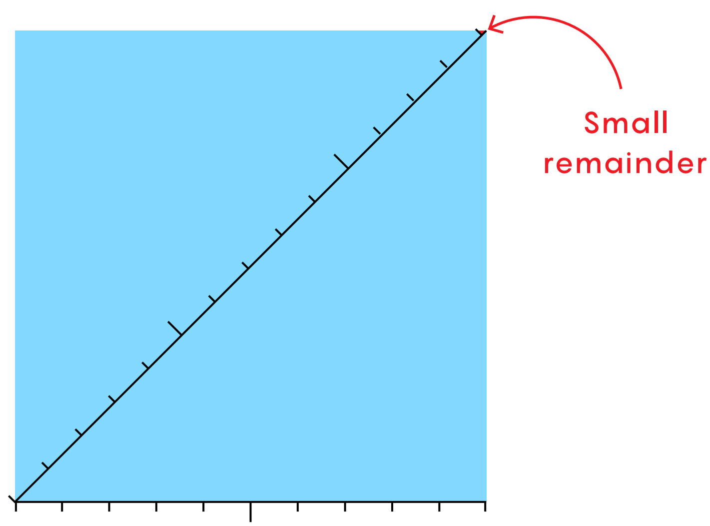
James O'Brien for *Quanta Magazine*

In 1886 the mathematician Leopold Kronecker famously said, "God Himself made the whole numbers — everything else is the work of men." Indeed, mathematicians have introduced new sets of numbers besides the ones used to count, and they have labored to understand their properties.

Although each type of number has its own fascinating and complicated history, today they are all so familiar that they are taught to schoolchildren. Integers are just the whole numbers, plus the negative whole numbers and zero. Rational numbers are those that can be expressed as a quotient of integers, such as 3, −1/2 and 57/22. Their decimal expansions either terminate (−1/2 = −0.5) or eventually repeat (57/22 = 2.509090909...). That means if a number has decimal digits that go on forever without repeating, it's irrational. Together the rational and irrational numbers comprise the real numbers. Advanced students learn about the complex numbers, which are formed by combining the real numbers and imaginary numbers; for instance, $i = \sqrt{-1}$.

One set of numbers, the transcendentals, is not as well known. Paradoxically, these numbers are both plentiful and exceedingly difficult to find. And their history is intertwined with a question that plagued mathematicians for millennia: Using only a compass and a straightedge, can you draw a square with the same area as a given circle? Known as squaring the circle, the question was answered only after the invention of algebra and a deeper understanding of $\pi$ — the ratio of the circumference of any circle to its diameter.

What does it mean to discover a new set of numbers? Today we say that Hippasus of Metapontum, who lived in approximately the fifth century BCE, discovered irrational numbers. In fact, his discovery was geometric, not arithmetic. He showed that it's possible to find two line segments, like the side and diagonal of a square, that can't be divided into parts of equal length. Today we would say that their lengths are not rational multiples of each other. Because the diagonal is $\sqrt{2}$ times as long as the side, $\sqrt{2}$ is irrational.




It is impossible to divide the side and diagonal of a square into parts of equal length. Here a length divides the side into 10 equal parts, but the diagonal is divided into 14 equal parts with a small remainder.

In terms of constructions possible with just a compass and straightedge — the mathematical tools of antiquity — if we begin with a unit-length line segment, it's possible to construct a segment with any positive rational length. However, we can also construct some irrational lengths. For instance, we've seen how to make $\sqrt{2}$; another famous irrational number, the golden ratio, $(1+\sqrt{5})/2$, is the diagonal of a regular pentagon with side length 1.

Roughly 2,000 years after the Greeks first posed the question of squaring the circle, René Descartes applied new algebraic techniques to show in his 1637 treatise *La Géométrie* that the constructible lengths are precisely those that can be expressed using integers and the operations of addition, subtraction, multiplication, division and the calculation of square roots. Notice that all positive rational numbers have this form, as do $\sqrt{2}$ and the golden ratio. If $\pi$ could be written in this way, it would finally let geometers square the circle — but $\pi$ was not so easy to classify.

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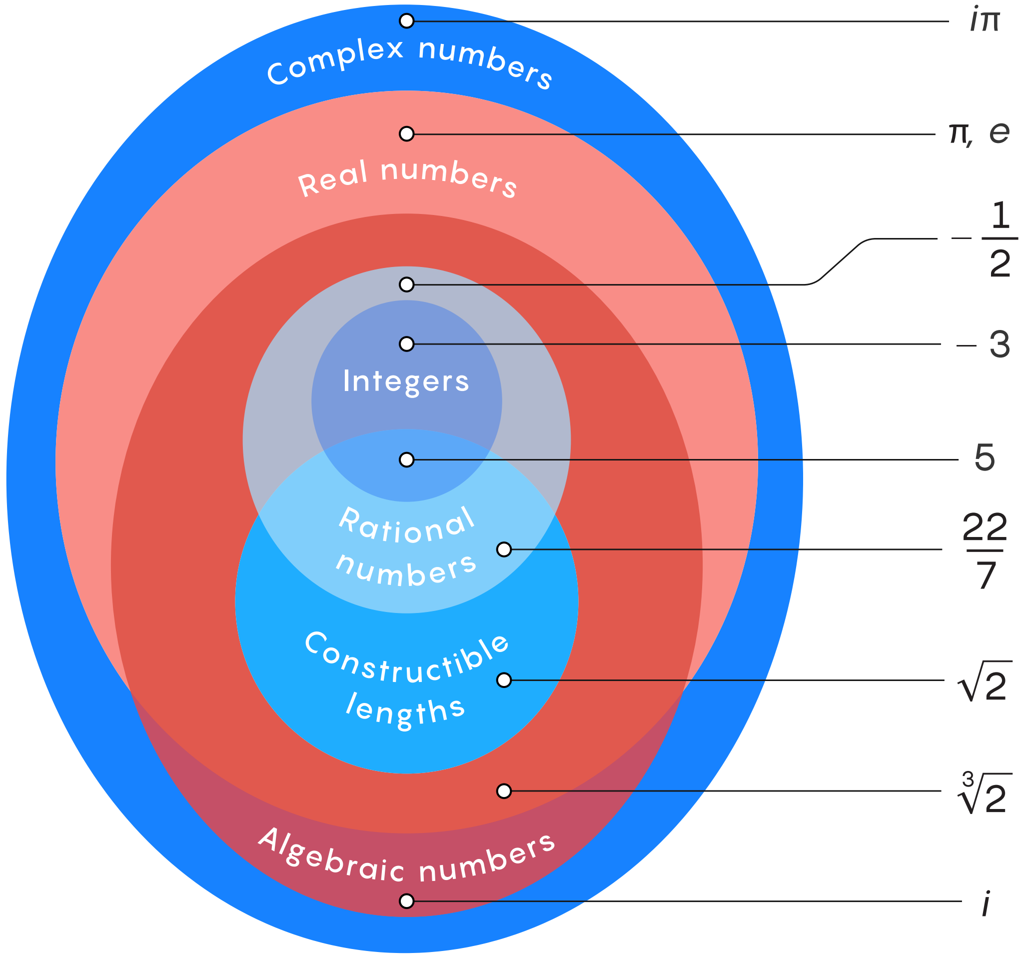
In the next 200 years, algebra matured significantly, and in 1837 a little-known French mathematician named Pierre Wantzel connected constructible numbers to polynomials — mathematical expressions that involve variables raised to various powers. In particular, he proved that if a length is constructible, then it must also be a root, or value that produces zero, of a certain type of polynomial, namely one that can't be factored, or simplified, further, and whose degree (the largest exponent of *x*) is a power of 2 (so 2, 4, 8, 16 and so on).

For instance, $\sqrt{2}$ and the golden ratio are constructible, and they are roots of the polynomials $x^2-2$ and $x^2-x-1$, respectively. On the other hand, $\sqrt[3]{2}$ is a root of the degree 3 polynomial $x^3-2$, which doesn't qualify, so it is impossible to construct a segment of this length.

Wantzel used his results to resolve other classical problems by proving that they can't be solved — it is impossible to trisect some angles, it is impossible to double the cube and it is impossible to construct certain regular polygons. But because the exact nature of $\pi$ remained a mystery, the question of squaring the circle remained open.

The key to resolving the problem, it turned out, was to cleverly divide the set of complex numbers into two sets, much as earlier generations partitioned the real numbers into rational and irrational numbers. Many complex numbers are the root of some polynomial with integer coefficients; mathematicians call these numbers algebraic. But this isn't true for all numbers, and these non-algebraic values are called transcendental.

Every rational number is algebraic, and some irrational numbers are too, like $\sqrt[3]{2}$. Even the imaginary number $i$ is algebraic, as it is a root of $x^2+1$.




This diagram shows the relationships between the various kinds of numbers. An irrational number is any real number that is not rational, and a transcendental number is any complex number that is not algebraic.

It was not obvious that transcendental numbers should exist. Moreover, it's challenging to prove that a given number is transcendental because it requires proving a negative: that it is not the root of any polynomial with integer coefficients.

In 1844, Joseph Liouville found the first one by coming at the problem indirectly. He discovered that irrational algebraic numbers cannot be approximated well by rational numbers. So if he could find a number that was approximated well by fractions with small denominators, it would have to be something else: a transcendental number. He then constructed just such a number.

Liouville's manufactured number,

$L = 0.110001000000000000000001 0\ldots,$

contains only 0s and 1s, with the 1s occurring in certain designated places: the values of $n!$. So the first 1 is in the first (1!) place, the second is in the second (2!) place, the third is in the sixth (3!) place, and so on. Notice that as a result of his careful construction, 1/10, 11/100, and 110,001/1,000,000 are all very good approximations of *L* — better than one would expect given the size of their denominators. For instance, the third of these values has 3! (six) decimal digits, 0.110001, but agrees with *L* for a total of 23 digits, or $4! - 1$.

Despite *L* proving that transcendental numbers exist, π does not satisfy Liouville's criterion (it can't be well approximated by rational numbers), so its classification remained elusive.

The key breakthrough occurred in 1873, when Charles Hermite devised an ingenious technique to prove that *e*, the base of the natural logarithm, is transcendental. This was the first non-contrived transcendental number, and nine years later it allowed Ferdinand von Lindemann to extend Hermite's technique to prove that π is transcendental. In fact he went further, showing that $e^d$ is transcendental whenever *d* is a nonzero algebraic number. Rephrased, this says that if $e^d$ is algebraic, then *d* is either zero or transcendental.

To prove that π is transcendental, Lindemann then made use of what many people view as the most beautiful formula in all of mathematics, Euler's identity: $e^{\pi i} = -1$. Because -1 is algebraic, Lindemann's theorem states that $\pi i$ is transcendental. And because $i$ is algebraic, π must be transcendental. Thus, a segment of length π is impossible to construct, and it is therefore impossible to square the circle.

Although Lindemann's result was the end of one story, it was just an early chapter in the story of transcendental numbers. Much still had to be done, especially, as we'll see, given how prevalent these misfit numbers are.

Shortly after Hermite proved that *e* was transcendental, Georg Cantor proved that infinity comes in different sizes. The infinity of rational numbers is the same as the infinity of whole numbers. Such sets are called countably infinite. However, the sets of real numbers and irrational numbers are larger; in a sense that Cantor made precise, they are "uncountably" infinite. In the same paper, Cantor proved that although the set of algebraic numbers contains all rational numbers and infinitely many irrational numbers, it is still the smaller, countable size of infinity. Thus, its complement, the transcendental numbers, is uncountably infinite. In other words, the vast majority of real and complex numbers are transcendental.

Yet even by the turn of the 20th century, mathematicians could conclusively identify only a few. In 1900, David Hilbert, one of the most esteemed mathematicians of the era, produced a now-famous list of the 23 most important unsolved problems in mathematics. His seventh problem, which he considered one of the harder ones, was to prove that $a^b$ is transcendental when $a$ is algebraic and not equal to zero or 1, and $b$ is an algebraic irrational number.

In 1929, the young Russian mathematician Aleksandr Gelfond proved the special case in which $b = \pm i\sqrt{r}$ and $r$ is a positive rational number. This also implies that $e^{\pi}$ is transcendental, which is surprising because neither $e$ nor $\pi$ is algebraic, as required by the theorem. However, by cleverly manipulating Euler's identity again, we see that

$$e^{\pi} = e^{-i\pi i} = (e^{\pi i})^{-i} = (-1)^{-i}.$$

Shortly afterward, Carl Siegel extended Gelfond's proof to include values of $b$ that are real quadratic irrational numbers, allowing him to conclude that $2^{\sqrt{2}}$ is transcendental. In 1934, Gelfond and Theodor Schneider independently solved the entirety of Hilbert's problem.

Work on transcendental number theory continued. In the mid-1960s Alan Baker produced a series of articles generalizing the results of Hermite, Lindemann, Gelfond, Schneider and others, giving a much deeper understanding of algebraic and transcendental numbers, and for his efforts he received the Fields Medal in 1970, at age 31. One consequence of this work was proving that certain products, like $2^{\sqrt{2}} \times 2^{\sqrt[3]{2}}$ and $2^{\sqrt{2}} \times 2^{\sqrt{3}}$, are transcendental. Besides expanding our understanding of the numbers themselves, his work also has applications throughout number theory.

Today, open problems about transcendental numbers abound, and there are many specific, very transcendental-looking numbers whose classification remains unknown: $e\pi$, $e + \pi$, $e^{e}$, $\pi^{\pi}$ and $\pi^{e}$, to name a few. Just as the mathematician Edward Titchmarsh said of the irrationality of $\pi$, it may be of no practical use to know that these numbers are transcendental, but if we can know, it surely would be intolerable not to know.