TOPOLOGY

An Old Conjecture Falls, Making Spheres a Lot More Complicated

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The telescope conjecture gave mathematicians a handle on ways to map one sphere to another. Now that it has been disproved, the universe of shapes has exploded.

Samuel Velasco/*Quanta Magazine*

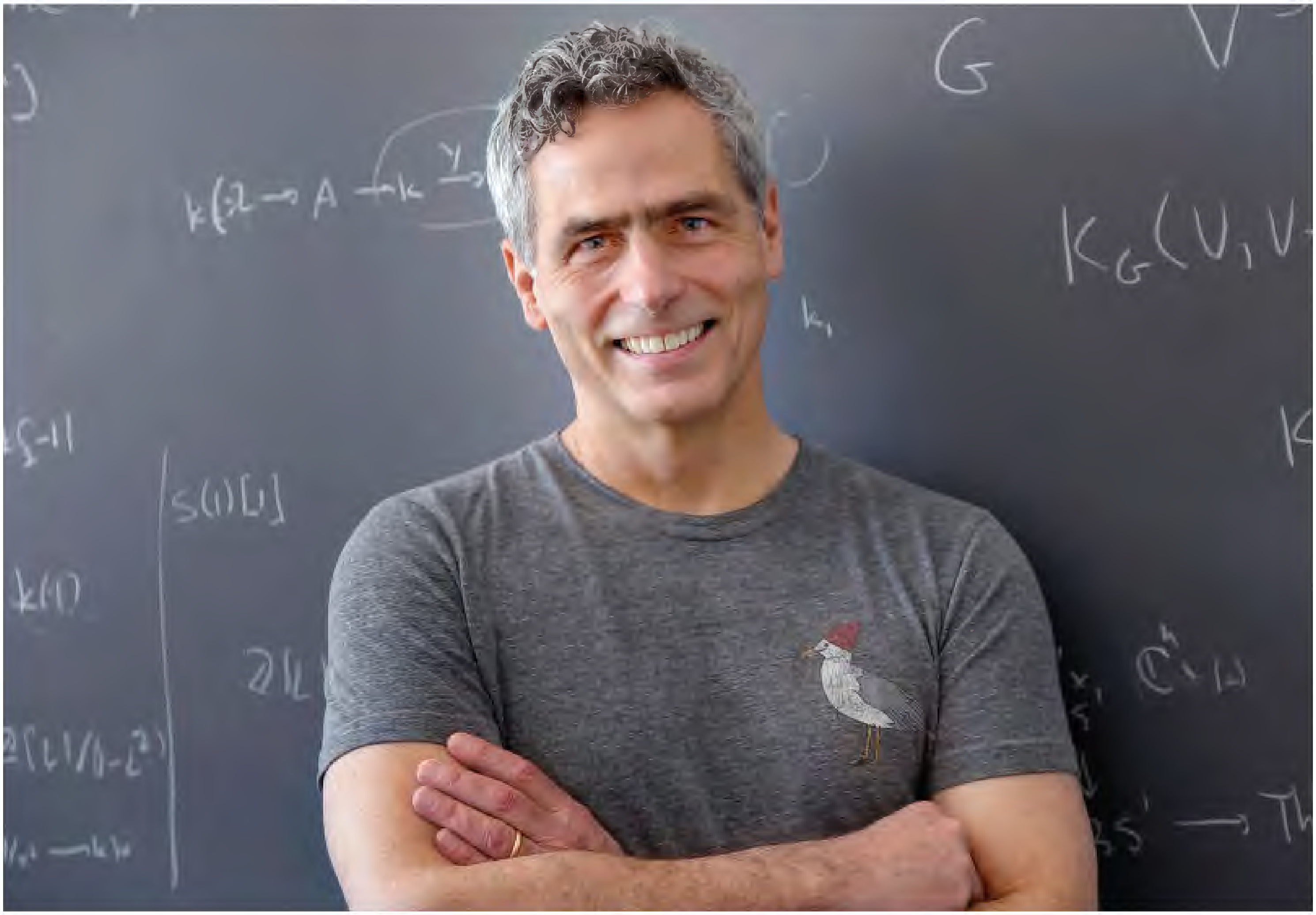
In early June, buzz built as mathematicians landed at London's Heathrow Airport. Their destination was the University of Oxford and a conference in honor of the 65th birthday of Michael Hopkins, a mathematician at Harvard University who'd served as a mentor to many of the attendees.

Hopkins made a name for himself in the late 1980s for work on seven conjectures that Doug Ravenel of the University of Rochester had formulated a decade earlier. They had to do with techniques for determining when two shapes, or spaces, that might look different are really the same. Hopkins and his collaborators proved all of Ravenel's conjectures save one, a problem with a suggestive but mysterious name called the telescope conjecture.

At the time, Hopkins laid his work on Ravenel's conjectures to rest. For decades afterward, the telescope conjecture seemed all but impossible to solve.

"You couldn't touch a theorem like that," Hopkins said.

But as mathematicians landed in London, there were rumors that it had been done — by a group of four mathematicians with ties to the Massachusetts Institute of Technology, three of whom had been advised by Hopkins in graduate school. The youngest of the four, a graduate student named Ishan Levy, was scheduled to give a talk on Tuesday, the second day of the conference, which seemed to be when a proof might be announced.



For his 65th birthday, Mike Hopkins’ students gave him proof that the telescope conjecture is false.

Courtesy of Mike Hopkins

"I had heard rumors that this was coming up, and I didn't know exactly what to expect," said Vesna Stojanoska, a mathematician at the University of Illinois, Urbana-Champaign who attended the conference.

It was soon clear the rumors were true. Beginning on Tuesday, and over the next three days, Levy and his co-authors — Robert Burklund, Jeremy Hahn and Tomer Schlank — explained to the crowd of some 200 mathematicians how they'd proved that the telescope conjecture was false, making it the only one of Ravenel's original conjectures not to be true.

The disproof of the telescope conjecture has wide-ranging implications, but one of the simplest and most profound is this: It means that in very high dimensions (think of a 100-dimensional sphere), the universe of different shapes is far more complicated than mathematicians anticipated.

Mapping the Maps

To classify shapes, or topological spaces, mathematicians distinguish between differences that matter and those that don't. Homotopy theory is a perspective from which to make those distinctions. It considers a ball and an egg to be fundamentally the same topological space, because you can bend and stretch one into the other without ripping either. In the same way, homotopy theory considers a ball and an inner tube to be fundamentally different because you have to tear a hole in the ball to deform it into the inner tube.

Homotopy is useful for classifying topological spaces — creating a chart of all the kinds of shapes that are possible. It's also important for understanding something else mathematicians care about: maps between spaces. If you have two topological spaces, one way to probe their properties is to look for functions that convert, or map, points on one to points on the other — input a point on space A, get a point on space B as your output, and do that for all the points on A.

To see how these maps work, and why they illuminate properties of the spaces involved, start with a circle. Now map it onto the two-dimensional sphere, which is the surface of a ball. There are infinitely many ways of doing this. If you imagine the sphere as Earth's surface, you could put your circle at any line of latitude, for example. From the perspective of homotopy theory, they're all equivalent, or homotopic, because they can all shrink down to a point at the north or south pole.

Next, map the circle onto the two-dimensional surface of an inner tube (a one-holed torus). Again, there are infinitely many ways of doing this, and most are homotopic. But not all of them. You could place a circle horizontally or vertically around the torus, and neither can be smoothly deformed into the other. These are two (of many) ways of mapping a circle onto the torus, while there is just one way to map it onto a sphere, reflecting a fundamental difference between the two spaces: The torus has one hole while the sphere has none.

It's easy to count the ways we can map from the circle to the two-dimensional sphere or torus. They're familiar spaces that are easy to visualize. But counting maps is much harder when higher-dimensional spaces are involved.

Dimensional Differences

If two spheres have the same dimension, there are always infinitely many maps between them. And if the space you're mapping from is lower-dimensional than the space you're mapping to (as in our example of the one-dimensional circle mapped onto a two-dimensional sphere), there is always only one map.

Partly for that reason, counting maps is most interesting when the space you're mapping from has a higher dimension than the space you're mapping to, like when you map a seven-dimensional sphere onto a three-dimensional sphere. In cases like those, the number of maps is always finite.

"The maps between spheres in general tend to be more interesting when the source has a larger dimension," Hahn said.

Moreover, the number of maps depends only on the difference in the number of dimensions (once the dimensions get big enough compared to the difference). That is, the number of maps from a 73-dimensional sphere to a 53-dimensional sphere is the same as the number of maps from a 225-dimensional sphere to a 205-dimensional sphere, because in both cases, the difference in dimension is 20.

Mathematicians would like to know the number of maps between spaces of any difference in dimension. They've managed to compute the number of maps for almost all differences in dimension up to 100: There are 24 maps between spheres when the difference is 20, and 3,144,960 when it's 23.

Mathematicians Ishan Levy, Robert Burklund, Jeremy Hahn, and Tomer Schlank (from left to right) found that the world of high-dimensional spheres gets very complicated very quickly.

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Archives of the Mathematisches Forschungsinstitut Oberwolfach (left); Jim Hoyer/UCPH; Christopher Harting; Archives of the Mathematisches Forschungsinstitut Oberwolfach

But calculating the number of maps for any difference larger than 100 exhausts modern computing power. And at the same time, mathematicians have not detected enough patterns in the number of maps to extrapolate further. Their goal is to fill out a table that specifies the number of maps for any difference in dimension, but that goal feels very far off.

"This is not a question I expect a complete solution to in the lifetime of my grandchildren," said Ravenel, who is 76.

The telescope conjecture makes a prediction about how the number of maps grows as the difference in dimension increases. In effect, it predicts that the number grows slowly. If it had been true, it would have made the problem of filling out that table a little bit easier.

Doubt Into Disbelief

The telescope conjecture got its name in an improbable way.

It started from the fact that in very high dimensions, geometric intuition formed in lower dimensions often breaks down, and it's difficult to count maps between spheres. But in formulating his conjecture, Ravenel understood that you don't have to. Instead of counting maps between spheres, you can make an easier proxy count of maps between spheres and objects called telescopes.

Telescopes involve a series of copies of a closed higher-dimensional curve, each one a scaled-down version of the one that came before it. The series of curves resembles the interlocking tubes of an actual collapsible telescope. "As bizarre as this telescope sounds when you describe it, it's actually an easier object to deal with than the sphere itself," Ravenel said.

But still, spheres can map onto telescopes in many different ways, and the challenge is knowing when those maps are genuinely distinct.

To determine whether two spaces are homotopic requires a mathematical test known as an invariant, which is a calculation based on properties of the spaces. If the calculation yields a different value for each space, you know they're unique from the perspective of homotopy.

There are many kinds of invariants, and some can perceive differences that other invariants are blind to. The telescope conjecture predicts that an invariant called Morava *E*-theory (and its symmetries) can perfectly distinguish all maps between spheres and telescopes up to homotopy — that is, if Morava *E*-theory says the maps are distinct, they're distinct, and if it says they're the same, they're the same.

But by 1989 Ravenel had begun to doubt it was true. His skepticism emerged from calculations he performed that did not seem to be consistent with the conjecture. But it wasn't until October of that year, when a massive earthquake struck the Bay Area while he was in Berkeley, that those doubts codified into full-fledged disbelief.

“I came to this conclusion within a day or two of the earthquake, so I like to think something happened that made me think it wasn’t true,” said Doug Ravenel.

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Archives of the Mathematisches Forschungsinstitut Oberwolfach

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Disproving the telescope conjecture would require finding a more powerful invariant that could see things Morava *E*-theory cannot. For decades no such invariant seemed to be available, placing the conjecture firmly out of reach. But progress in recent years changed that — and Burklund, Hahn, Levy and Schlank capitalized on it.

The Exploding Exotic

Their proof relies on a set of tools called algebraic *K*-theory, which was established in the 1950s by Alexander Grothendieck and has developed rapidly over the last decade. It has applications across mathematics, including in geometry, where it has the ability to supercharge an invariant.

The four authors use algebraic *K*-theory as a gadget: They input Morava *E*-theory, and their output is a new invariant that they refer to as the algebraic *K*-theory of the fixed points of Morava *E*-theory. They then apply this new invariant to maps from spheres to telescopes and prove that it can see maps that Morava *E*-theory cannot.

And it's not just that this new invariant sees a few more maps. It sees many more, even infinitely more. So many more that it's fair to say Morava *E*-theory was barely scratching the surface when it came to identifying maps from spheres to telescopes.

Infinitely more maps from spheres to telescopes means infinitely more maps between spheres themselves. The number of such maps is finite for any difference in dimension, but the new proof shows that the number grows quickly and inexorably.

That there are so many maps points to an unsettling geometric reality: There are so many spheres.

In 1956 John Milnor identified the first examples of what are called "exotic" spheres. These are spaces that can be deformed into the actual sphere from the perspective of homotopy but are different from the sphere in a certain precise sense. Exotic spheres don't exist at all in dimensions one, two or three, and no one has discovered examples of them below dimension seven — the dimension where Milnor first found them. But as the dimension grows, the number of exotic spheres explodes. There are 16,256 in dimension 15, and 523,264 in dimension 19.

And yet, as huge as those numbers are, the disproof of the telescope conjecture means there are many, many more. The disproof means there are more maps between spheres than anticipated back when Ravenel stated the conjecture, and the only way you get more maps is by having a greater variety of spheres to map between.

There are different types of progress in math and science. One kind brings order to chaos. But another intensifies the chaos by dispelling hopeful assumptions that weren't true. The disproof of the telescope conjecture is like that. It deepens the complexity of geometry and raises the odds that many generations of grandchildren will come and go before anyone fully understands maps between spheres.

“Every major advance in the subject seems to tell us the answer is a lot more complicated than we thought before,” Ravenel said.